1 Questions:

10.6 #19: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{n+1}}{\sqrt{(n+1)^2+1}} \frac{\sqrt{n^2+1}}{(-1)^n x^n} \right| &= \lim_{n\to\infty} \left| \frac{x \cdot x^n \sqrt{n^2+1}}{(\sqrt{(n+1)^2+1})x^n} \right| \\
&= |x| \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{\sqrt{(n+1)^2+1}} \\
&= |x| < 1
\end{aligned}$$

Therefore the series converges absolutely for $-1 < x < 1$. Check endpoints: $x = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$$

converegs by the alternating series test: check $\frac{1}{\sqrt{n^2+1}}$ decreasing? Yes because the denominator is increasing. Check: $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$. Therefore the above series converges. So we include $x = 1$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n^2+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

Limit compare to $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2}}$:

$$\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/\sqrt{n^2}} = \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2+1}} = 1.$$

Thus both series must diverge because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}}$ diverges. So we do not include $x = -1$. Hence the interval of convergence is $(-1, 1]$.

2 Examples:

1. Find the first three terms of the Taylor series for $2\sin(x)\cos(x)$. Let $f(x) = 2\sin(x)\cos(x) = \sin(2x)$.

$$\begin{aligned}
f(x) &= \sin(2x) & f(0) &= 0 \\
f'(x) &= 2\cos(2x) & f'(0) &= 2 \\
f''(x) &= -4\sin(2x) & f''(0) &= 0 \\
f^{(3)}(x) &= -8\cos(2x) & f^{(3)}(0) &= -8 \\
f^{(4)}(x) &= 16\sin(2x) & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= 32\cos(2x) & f^{(5)}(0) &= 32 = 2^5 \\
& & f^{(2n+1)}(0) &= (-1)^n 2^{2n+1}
\end{aligned}$$

Thefore the Mac series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$$

2. Use the Maclaurin series for $\ln(x+1)$ to approximate $\ln(4)$ (Note: $\ln(4) = -\ln(1/4)$).

3. Find the Taylor series for $\sin(x)$ about $\pi/2$. Use this expansion to approximate $\sin(1)$.

3: To find a Taylor series, let $f(x) = \sin(x)$.

$$f(x) = \sin(x) \quad f(\pi/2) = 1 \qquad f'(x) = \cos(x) \quad f'(\pi/2) = 0$$
$$f''(x) = -\sin(x) \quad f''(\pi/2) = -1 \qquad f^{(3)}(x) = -\cos(x) \quad f^{(3)}(\pi/2) = 0$$
$$f^{(4)}(x) = \sin(x) \quad f^{(4)}(\pi/2) = 1 \qquad f^{(5)}(x) = \cos(x) \quad f^{(5)}(\pi/2) = 0$$
$$f^{(2n)}(x) = (-1)^n \sin(x) \quad f^{(2n)}(\pi/2) = (-1)^n \qquad f^{(2n+1)} = (-1)^n \cos(x) \quad f^{(2n+1)}(\pi/2) = 0$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x - \pi/2)^{2n} = \sin(x)$$

To approximate $\sin(1) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(1 - \pi/2)^{2n}$

3 Group Problems:

1. Find a Taylor series for $f(x) = e^x$ centered at $a = 3$.
2. Find a Taylor series for $f(x) = \sin(x)$ centered at $\pi/2$.
3. Find a Maclaurin series for $f(x) = e^{2x}$.
4. Use an infinite series to evaluate the integral $\displaystyle\int_0^{\pi} \sin(x^2)\,dx$.