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CONVOLUTION OF THE IMPACT THREE-DIMENSIONAL ELASTO-DYNAMICS AND DYNAMIC STRESS INTENSITY FACTOR FOR AN ELLIPTIC CRACK*

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ABSTRACT: This paper presents a formulation for three-dimensional elasto-dynamics with an elliptic crack based on the Laplace and Fourier transforms and the convolution theorem. The dynamic stress intensity factor for the crack is determined by solving a Fredholm integral equation of the first kind. The results of this paper are very close to those given by the two-dimensional dual integral equation method.

KEY WORDS: dynamic loading, three-dimensional elliptic crack, dynamic stress intensity factor

1 INTRODUCTION

The three-dimensional elliptic crack under impact loading is one of the important problems of dynamic fracture. Due to the complexity of its mathematical treatment, there are few analytic solutions in this respect published. References [1~3] developed a method for solving three-dimensional elasto-dynamics, in which some solutions with respect to the dynamic contact were given. The fundamental difficulty of the three-dimensional dynamic crack problem lies in the solution of two-dimensional dual integral equations. In the present work, on the basis of the formulation proposed by Ref.[2] and using the convolution theorem of the Fourier transform, the crack problem is reduced to the solution of a Fredholm integral equation of the first kind. This method avoids the difficulty in solving the two-dimensional dual integral equations, and the results show certain reasonability.

2 FUNDAMENTAL EQUATIONS

The three-dimensional elasto-dynamic problem is governed by the equations

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \qquad \nabla^2\boldsymbol{\psi} = \frac{1}{c_2^2}\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} \tag{1}$$

in which ϕ and $\boldsymbol{\psi}$ are Lamé scalar potential and vector potential, respectively, defined by

$$\boldsymbol{u} = \nabla\phi + \nabla\times\boldsymbol{\psi} \tag{2}$$

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$\boldsymbol{u}$ denotes displacement vector, ∇ and $\nabla\times$ are gradient and curl operators, c_1 and c_2 are the propagating velocities of longitudinal and transverse waves

$$c_1 = [(\lambda + 2\mu)/\rho]^{1/2} \qquad c_2 = (\mu/\rho)^{1/2} \tag{3}$$

λ, μ and ρ the Lamé constants and mass density of the material.

Reference [2] provided the integral expression of the solution of Eqs.(1) in terms of the Laplace transform

$$f^*(x_1, x_2, x_3, s) = \int_0^\infty f(x_1, x_2, x_3, t)\mathrm{e}^{-st}\mathrm{d}t \tag{4}$$

and the double Fourier transform

$$\tilde{f}^*(\xi_1, \xi_2, x_3, s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x_1, x_2, x_3, s)\mathrm{e}^{\mathrm{i}(\xi_1 x_1 + \xi_2 x_2)}\mathrm{d}x_1\mathrm{d}x_2 \tag{5}$$

in which $f(x_1, x_2, x_3, t)$ represents $\phi(x_1, x_2, x_3, t)$ or any component of $\psi(x_1, x_2, x_3, t)$, x_i the spatial coordinate, t the time, s the Laplace transform parameter, ξ_1, ξ_2 the Fourior transform parameters, $\mathrm{i} = \sqrt{-1}$.

All field variables (e.g. stresses and displacements) can be expressed by the Laplace-Fourior transforms of the Lamé potentials through the generalized Hooke's law. We found that there is a very simple relation between the Laplace-Fourior transform of the normal stress σ_{33} and the normal displacement u_3 at plane $x_3 = 0$ as follows

$$\tilde{\sigma}_{33}^*(\xi_1, \xi_2, 0, s) = -\tilde{u}_3^*(\xi_1, \xi_2, 0, s)/\tilde{H}_{33}^*(\xi_1, \xi_2, s) \tag{6}$$

where

$$\begin{aligned}\tilde{H}_{33}^*(\xi_1, \xi_2, s) = -\gamma_1\left(\xi_1^2 + \xi_2^2 - \gamma_2^2\right)/2\pi\left\{\lambda\left(\gamma_1^2 - \xi_1^2 - \xi_2^2\right)\left(\gamma_2^2 + \xi_1^2 + \xi_2^2\right) + \right. \\ \left. 2\mu\left[\gamma_1^2\left(\gamma_2^2 + \xi_1^2 + \xi_2^2\right) - 2\gamma_1\gamma_2\left(\xi_1^2 + \xi_2^2\right)\right]\right\}\end{aligned} \tag{7}$$

is one of the components of the so-called generalized influence function, and

$$\gamma_1 = \left(\xi_1^2 + \xi_2^2 + \frac{s^2}{c_1^2}\right)^{1/2} \qquad \gamma_2 = \left(\xi_1^2 + \xi_2^2 + \frac{s^2}{c_2^2}\right)^{1/2} \tag{8}$$

formulae (6) and (7) are the basis of our following discussion.

3 THREE-DIMENSIONAL ELLIPTIC CRACK UNDER IMPACT LOADING

Three-dimensional elliptic crack under impact loading (see Fig.1) is one of the most important configurations of a dynamic three-dimensional embedded crack, in both theory and practice. The problem is governed by Eq.(1) coupled with the initial and boundary conditions as follows

$$\begin{aligned} & u_i(x_1, x_2, x_3, 0) = 0 \\ & \left.\frac{\partial u_i(x_1, x_2, x_3, t)}{\partial t}\right|_{t=0} = 0 \end{aligned} \tag{9}$$

$$\begin{aligned} &\sqrt{x_1^2+x_2^2+x_3^2}\to\infty: && \sigma_{33}=p_0 f(t) \qquad \text{other } \sigma_{ij}=0 \\ &x_3=0 \quad (x_1,x_2)\in\Omega: && \sigma_{33}=\sigma_{31}=\sigma_{32}=0 \\ &x_3=0 \quad (x_1,x_2)\notin\Omega: && u_3=0 \quad \sigma_{31}=\sigma_{32}=0 \end{aligned} \tag{10}$$

in which Ω represents the region occupied by the crack, σ_{ij} the stress tensor, p_0 a constant, $p_0 f(t)$ denotes remote tensile loading, $f(t)$ a function of time. The boundary conditions (10) can be replaced by the following equivalent conditions (they are equivalent to each other in the sense of Griffith theory)

$$\begin{aligned} &\sqrt{x_1^2+x_2^2+x_3^2}\to\infty: && \sigma_{ij}=0 \\ &x_3=0 \quad (x_1,x_2)\in\Omega: && \sigma_{33}=-p_0 f(t) \quad \sigma_{31}=\sigma_{32}=0 \\ &x_3=0 \quad (x_1,x_2)\notin\Omega: && u_3=0 \quad \sigma_{31}=\sigma_{32}=0 \end{aligned} \tag{11}$$

Because (9), (11) are equivalent with (9), (10), and are easy to be solved, we use (9), (11) in the following discussion.

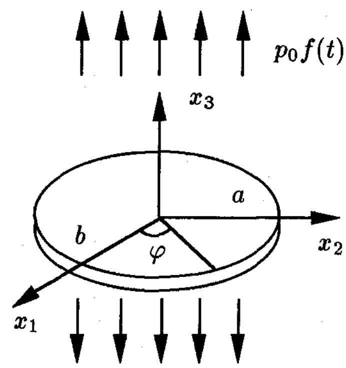


Fig.1 Three-dimensional elliptic crack under impact loading

4 CONVOLUTION AND THE FIRST KIND FREDHOLM INTEGRAL EQUATION

Performing the inversion of the double Fourier transform, we get

$$\sigma_{33}^*(x_1,x_2,0,s)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\tilde{u}_3^*(\xi_1,\xi_2,0,s)/\tilde{H}_{33}^*(\xi_1,\xi_2,s)\right\}\mathrm{e}^{-\mathrm{i}(\xi_1x_1+\xi_2x_2)}\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{12}$$

With the convolution theorem of the Fourier transform, it is rewritten as

$$\sigma_{33}^*(x_1,x_2,0,s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{u_3^*(\xi_1,\xi_2,0,s)h_{33}^*(x_1-\eta_1,x_2-\eta_2,s)\}\,\mathrm{d}\eta_1\mathrm{d}\eta_2 \tag{13}$$

where

$$u_{33}^*(x_1,x_2,0,s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{u}_3^*(\xi_1,\xi_2,0,s)\mathrm{e}^{-\mathrm{i}(\xi_1x_1+\xi_2x_2)}\mathrm{d}\xi_1\mathrm{d}\xi_2$$

$$h_{33}^*(x_1,x_2,0,s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1/\tilde{H}_{33}^*(\xi_1,\xi_2,s))\mathrm{e}^{-\mathrm{i}(\xi_1x_1+\xi_2x_2)}\mathrm{d}\xi_1\mathrm{d}\xi_2$$

Substituting the Laplace transform into the second formula of the boundary condition (11) yields

$$\sigma_{33}^*(x_1,x_2,0,s)=-p_0f^*(s)$$

where $f^*(s)$ is the Laplace transform of function $f(t)$. Comparing this formula with formula (13), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{u_3^*(\eta_1,\eta_2,0,s)h_{33}^*(x_1-\eta_1,x_2-\eta_2,s)\}\,\mathrm{d}\eta_1\mathrm{d}\eta_2=-p_0f^*(s) \qquad (14)$$

and from the third formula of (11),we know that

$$u_3(x_1,x_2,0,t)=0 \qquad (x_1,x_2)\notin\Omega$$

Then formula (14) is simplified as

$$\iint_{\Omega}\{u_3^*(\eta_1,\eta_2,0,s)h_{33}^*(x_1-\eta_1,x_2-\eta_2,s)\}\,\mathrm{d}\eta_1\mathrm{d}\eta_2=-p_0f^*(s) \qquad (14')$$

This is a two-dimensional Fredholm integral equation of the first kind in which $u_3^*(\eta_1, \eta_2,0,s)$ is unknown. From this equation, we can determine displacement $u_3^*(x_1,x_2,0,s)((x_1, x_2)\in\Omega)$, i.e., the crack opening displacement in the Laplace transform domain is determined, so is the corresponding dynamic stress intensity factor.

5 THE SOLUTION OF THE INTEGRAL EQUATION (14′)

The unknown function $u_3^*(\eta_1,\eta_2,0,s)$ can be expressed by the following series

$$u_3^*(\eta_1,\eta_2,0,s)=\sum_{m,n=0}^{\infty}C_{m,n}(s)\mathrm{J}_m(a\eta_1)\mathrm{J}_n(b\eta_2) \qquad (15)$$

where $C_{m,n}(s)$ are the coefficients to be determined, $\mathrm{J}_m(x)$ and $\mathrm{J}_n(x)$ the Bessel functions of the first kind with order m and n . Substituting (15) into (14′), we obtain the algebraic equation system for the unknown coefficients. The solution of the algebraic equation system determines the coefficients $C_{m,n}$, and the crack opening displacement at the Laplace transform domain.

6 DYNAMIC STRESS INTENSITY FACTOR

We rewrite the normal displacement $u_3^*(x_1,x_2,0,s)$ at the vicinity of the crack tip with distance ε measured from the tip as $u_3^*(\varepsilon,\varphi,0,s)$, where $\varphi=\mathrm{tg}^{-1}(x_2/x_1)$. Because the stress state at the crack tip is in a plane strain condition, we have the formula

$$u_3^*(\varepsilon,\varphi,0,s)=\frac{2(1-\nu^2)}{E\sqrt{\pi}}(2\varepsilon)^{1/2}K_{\mathrm{I}}^*(b/a,\varphi,s) \qquad (16)$$

where E and ν are Young's modulus and Poisson's ratio, K_{I}^* denotes the dynamic stress intensity factor of model I in the Laplace transform domain. Carrying out extrapolation with the formula (16) (let $\varepsilon \to 0$), we can calculates $K_{\rm I}^*$ from u_3^*, which is a function of the geometry parameters $b/a, \varphi$ and the Laplace transform parameters. Then taking the inversion of Laplace transform, we find that

$$K_{\rm I}(b/a, \varphi, t) = L^{-1}(K_{\rm I}^*(b/a, \varphi, s)) \tag{17}$$

This is the dynamic stress intensity factor in the physical time-space. The numerical inversion of the Laplace transform was systematically studied by one of the authors[4].

We adopt the normalized dynamic stress intensity factor

$$K_{\rm I}(t)/K_{\rm I}^{\rm static} \tag{18}$$

versus the normalized time c_2t/a, in which

$$K_{\rm I}^{\rm static} = \frac{p_0\sqrt{\pi}}{E(k)}\left(\frac{b}{a}\right)^{1/2}\left(a^2\sin^2\varphi + b^2\cos^2\varphi\right) \tag{19}$$

is the static stress intensity factor (i.e., is the well-known Green-Sneddon solution[5]), $E(k)$ is the completely elliptic integral of the second kind $E(k) = \int_0^{\pi/2}\left(1 - k^2\sin^2\theta\right)^{1/2}{\rm d}\theta$, k represents the module $k = \left[\left(a^2 - b^2\right)/a^2\right]^{1/2}$ of the ellipse.

7 NUMERICAL EXAMPLES

Suppose

$$f(t) = H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

is the Heaviside function, we deal with two kinds of materials with Poisson's ratio ν = 0.29, 0.34, and the geometry parameter $b/a = 0.25$, 0.5, respectively. The calculated results are shown in Fig.2.

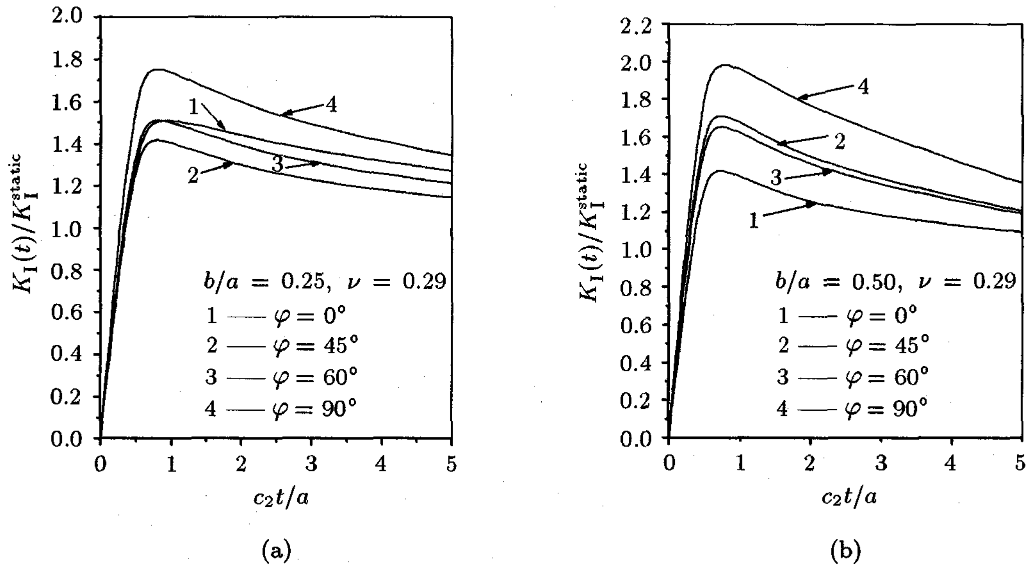
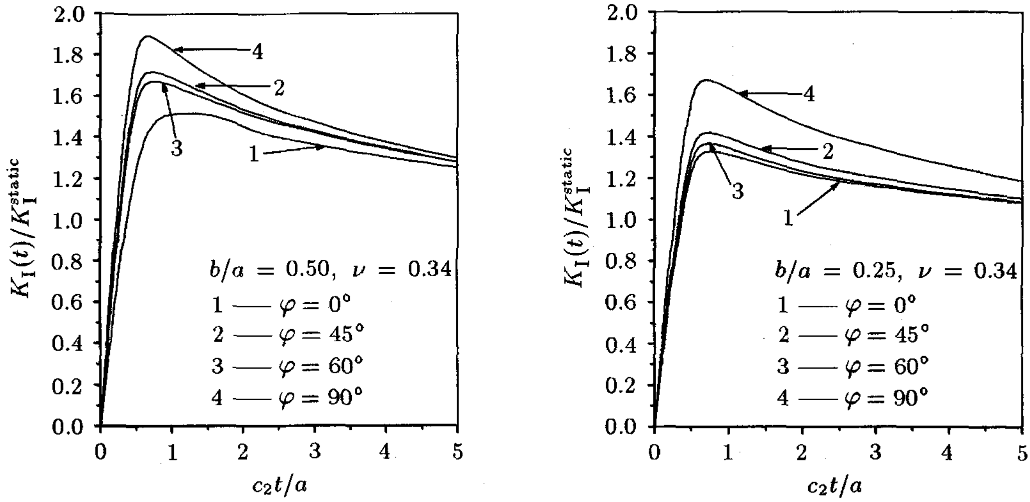


Fig.2 The normalized dynamic stress intensity factor versus time



(c) (d)

Fig.2 The normalized dynamic stress intensity factor versus time (continued)

8 CONCLUSIONS AND DISCUSSIONS

The results given in Fig.2 show that the dynamic stress intensity factors for different materials exhibit some similarity, in their time dependence. Therefore, the results can be used to describe the dynamic fracture behaviour of common structural materials.

Previously, dynamic solutions available are only for two-dimensional cracks or penny-shaped crack under axi-symmetrical impact loading[5,6], which may be regarded as special cases of the present work, i.e., the cases for $b/a \to 0$ or $b/a \to 1$. The comparison shows that the time variation of dynamic stress intensity factors for different configuration cracks has certain common characters. Of course, the effects of parameters b/a and φ are given in the present paper.

The present paper is mainly concerned with the impact fracture dynamics for a general three-dimensional crack. The aim of this study is to find out the early time behaviours of dynamic response of the material with crack or crack-like defect. Although we are still lack experimental results for dynamic behavior of three-dimensional elliptic crack, the comparison of our theoretical results with the well-known two-dimensional dynamic crack experiments indicates that the early time behaviour of the elliptic crack is reasonable. Secondly, the results given in Fig.2 also demonstrate that the transient dynamic trend approaches the static state with the increase of time.

The effect of the Poisson ratio of materials on the dynamic stress intensity factor is examined. It is well known that the Poisson ratio may vary considerably in some cases, and the influence of such variation is not so significant in general. But in certain special cases, e.g. in the earthquake, the variation of Poisson's ratio (which describes the variation of c_2/c_1) is quite important[6]. It can be regarded as a precursor parameter of earthquakes[7]. As a matter of fact, earthquake is a dynamic fracture process of the earth crust. The connection between fracture dynamics and earthquake rupture dynamics is important.

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