Problems for Preliminary Exam
Applied Mathematics
August 2022

Ordinary Differential Equations

1. Show that every integral curve of

$$\dot{x} = \sqrt[3]{\frac{x^2+1}{t^4+1}}, \quad x(t) \in \mathbb{R},$$

has two horizontal asymptotes.

2. Is there a bounded solution to (here $x(t) \in \mathbb{R}$)

$$\dot{x} - x = \cos t - \sin t?$$

3. Let

$$\dot{x} = f(x), \quad x(t) \in \mathbb{R}^n, \quad f \in C^1, \quad f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad f(\hat{x}) = 0, \quad \hat{x} \in \mathbb{R}^n.$$

Give a definition for $\hat{x}$ to be Lyapunov stable, asymptotically stable, or unstable. Formulate the definition for a differentiable $V\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ to be a Lyapunov function for $\hat{x}$, a strict Lyapunov function for $\hat{x}$.

Formulate and prove Lyapunov's theorem on (Lyapunov, asymptotic) stability of $\hat{x}$ by direct Lyapunov method (i.e., by assuming existence of a (strict) Lyapunov function).

4. Sketch the phase portrait of

$$\ddot{x} = 1 + 2\sin x, \quad x(t) \in \mathbb{R}.$$

5. For which α the trivial equilibrium of

$$\begin{aligned}
\dot{x}_1 &= \alpha x_1 - x_2, \\
\dot{x}_2 &= \alpha x_2 - x_3, \\
\dot{x}_3 &= \alpha x_3 - x_1,
\end{aligned}$$

is Lyapunov stable, asymptotically stable, unstable?

6. Find Green's function for

$$y'' + y = f(x), \quad y(0) = y(\pi), \quad y'(0) = y'(\pi).$$

7. Show that the problem

$$\dot{x} = 2\sqrt[3]{tx}, \quad x(0) = 0$$

has more than one solution passing through the origin. For the full credit describe *all* the solution to this problem.

Which condition(s) of the existence and uniqueness theorem fail?