Counterexamples Related to Rotations of Shadows of Convex Bodies

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ABSTRACT. We construct examples of two convex bodies K, L in $\mathbb{R}^n$, such that every projection of K onto an $(n-1)$-dimensional subspace can be rotated to be contained in the corresponding projection of L, but K itself cannot be rotated to be contained in L. We also find necessary conditions on $K, L \subset \mathbb{R}^3$ to ensure that K can be rotated to be contained in L if all the two-dimensional projections have this property.

1. INTRODUCTION

Let K be a convex body in $\mathbb{R}^n$. Given a unit vector $\xi \in S^{n-1}$, we denote by $K|\xi^\perp$ the orthogonal projection of K on the hyperplane $\xi^\perp = \{x \in \mathbb{R}^n \mid x \cdot \xi = 0\}$. Let $\mathrm{SO}(n)$ be the group of rotations in $\mathbb{R}^n$, and $\mathrm{SO}(n-1, \xi^\perp)$ be the group of rotations on the hyperplane $\xi^\perp$. In this paper, we study the following problem.

Problem 1.1. *Let K, L, be convex bodies in $\mathbb{R}^n$. Suppose that for every $\xi \in S^{n-1}$, the orthogonal projection $K|\xi^\perp$ can be rotated around the origin to fit inside $L|\xi^\perp$: that is, there exists a rotation $\varphi_\xi \in \mathrm{SO}(n-1, \xi^\perp)$ such that $\varphi_\xi(K|\xi^\perp) \subseteq L|\xi^\perp$. Then, we have the following questions:*

(a) *Does it follow that L contains a rotation of K; that is, is there a $\psi \in \mathrm{SO}(n)$ such that $\psi(K) \subseteq L$?*
(b) *Does it follow that* $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$*?*

In [7], D. Klain studied the same questions with translations instead of rotations. He proved that the answer to question 1.1 (a) for translations is negative in general, in any dimension. A counterexample is obtained by considering a ball B, together with the dilated simplex $(1+\varepsilon)T$, where T is the simplex inscribed in B. Then, for any $\varepsilon > 0$, the dilated simplex $(1+\varepsilon)T$ is not contained in the ball, nor can it be translated to fit inside; however, if ε is small enough, all the projections

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of $(1+\varepsilon)T$ on hyperplanes can be translated to fit inside the corresponding projections of the ball. Klain also proved that if both bodies are centrally symmetric, the answer to Problem 1.1 (a) for translations is affirmative.

Regarding question 1.1 (b) for translations, Klain showed that the answer is negative, in general, but that there exists a class of bodies such that the answer to 1.1 (b) for translations is affirmative if L belongs to that class.

Problem 1.1 and the analogous problem for translations are both related to the well-known Shephard's Problem (see [12]).

Problem 1.2 (Shephard's Problem). *Let K, L be origin symmetric convex bodies. If for every $\xi \in S^{n-1}$ we have $\mathrm{vol}_{n-1}(K|\xi^\perp) \le \mathrm{vol}_{n-1}(L|\xi^\perp)$, does it follow that $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$?*

It was proven independently by Petty [9] and Schneider [11] that the answer to Shephard's Problem is negative in general in dimension $n \ge 3$. In other words, a body K may have greater volume than another body L, even if all projections of K have smaller $(n-1)$-dimensional volume than the corresponding projections of L. In fact, K may be taken to be a ball, while L is a centrally symmetric double cone (see [3, Theorem 4.2.4]). Petty and Schneider also proved that the answer is affirmative under the additional assumption that the body L is a *projection body* (see [3, Section 4.1]).

Observe that if K and L are two convex bodies for which the answer to Problem 1.1 (a) is affirmative for either rotations or translations, then Shephard's problem for K and L also has an affirmative answer.

It is natural to consider the analogous question to Problem 1.1, replacing projections by sections. Here, $K \cap \xi^\perp$ denotes the section of K by the hyperplane $\xi^\perp$.

Problem 1.3. *Let K, L, be convex bodies in $\mathbb{R}^n$. Suppose that for every $\xi \in S^{n-1}$, the section $K \cap \xi^\perp$ can be rotated around the origin to fit inside $L \cap \xi^\perp$. We have the following questions:*

(a) *Does L contain a rotation of K?*
(b) *Does it follow that $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$?*

In the case of Problem 1.3 for translations, it is known that if K and L are bodies containing the origin in their interior, such that $K \cap \xi^\perp$ is a translate of $L \cap \xi^\perp$ for every $\xi \in S^{n-1}$, then K is a translate of L (see [3, Theorem 7.1.1]).

We note that Klain's and Shephard's counterexamples will not work for our Problems 1.1 and 1.3, since in both cases, one of the bodies they consider is a ball, which is invariant under rotations, as are all of its projections and sections. In Section 3, we present counterexamples to Problem 1.1 (a) and Problem 1.3 (a). The first counterexample is in $\mathbb{R}^3$, and consists of a cylinder C and a double cone K. We note that both the cylinder and double cone are centrally symmetric bodies, and hence, unlike the case of translations proved by Klain, our Problem 1.1 (a) does not have an affirmative answer for centrally symmetric bodies. The second example, which works in general dimension n, is given by appropriately chosen perturbations of two balls, following ideas of Kuzminykh and Nazarov. However, none of our counterexamples provide an affirmative answer to Problems 1.1 (b) or 1.3 (b).

In Section 4, we obtain a positive answer for Problem 1.1 (a), in $\mathbb{R}^3$, assuming a Hadwiger-type additional condition on the bodies K and L (see [6]). We also prove that the answer to Problem 1.3 (b) is affirmative (Theorem 4.1). However, for the case of projections, the argument only allows us to conclude the relation $\mathrm{vol}_n(K^*) \geq \mathrm{vol}_n(L^*)$ for the polar bodies, while Problem 1.1 (b) remains open.

In Section 5, we study the case in which the projections of K are *equal*, up to a rotation, to the corresponding projections of L. We prove that if K and L are bodies in $\mathbb{R}^3$ with countably many diameters, and that their hyperplane projections do not have certain rotational symmetries, then $K = \pm L$. For $n \geq 4$, an n-dimensional version of this result has been obtained by the authors and D. Ryabogin in [1], following ideas of Golubyatnikov [5] and Ryabogin [10].

2. Notation

In this section, we introduce the notation that will be used throughout the paper.

The unit sphere in $\mathbb{R}^n$ ($n \geq 2$) is S^{n-1}. The notation $SO(n)$ for the group of rotations in $\mathbb{R}^n$, and $SO(k)$, $2 \leq k \leq n$, for their subgroups is standard. If $\mathcal{U} \in SO(n)$ is an orthogonal matrix, we will write $\mathcal{U}^t$ for its transpose.

We write $\varphi_\xi \in SO(n-1, \xi^\perp)$, meaning that there exists a choice of an orthonormal basis in $\mathbb{R}^n$ and a rotation $\Phi \in SO(n)$, with a matrix written in this basis, such that the action of Φ on $\xi^\perp$ is the rotation φ_ξ in $\xi^\perp$, and the action of Φ in the ξ direction is trivial, that is, $\Phi(\xi) = \xi$.

We refer to [3, Chapter 1] for the next definitions. A *body* in $\mathbb{R}^n$ is a compact set which is equal to the closure of its non-empty interior. A *convex body* is a body K such that for every pair of points in K, the segment joining them is contained in K. For $x \in \mathbb{R}^n$, the *support function* of a convex body K is defined as $h_K(x) = \max\{x \cdot y \mid y \in K\}$ (see page 16 in [3]). The support function uniquely determines a convex body, and $h_{K_1} \leq h_{K_2}$ if and only if $K_1 \subseteq K_2$.

The *width function* $\omega_K(x)$ of K in the direction $x \in S^{n-1}$ is defined as $\omega_K(x) = h_K(x) + h_K(-x)$. The segment $[z, y] \subset K$ is called the *diameter* of the body K if $|z - y| = \max_{\{\theta \in S^{n-1}\}} \omega_K(\theta)$. Note that a convex body K can have at most one diameter parallel to a given direction, for if K has two parallel diameters d_1 and d_2, then K contains a parallelogram with sides d_1 and d_2, and one of the diagonals of this parallelogram is longer than d_1. If a diameter of K is parallel to the direction $\zeta \in S^{n-1}$, we will denote it by $d_K(\zeta)$. We say that a convex body $K \subset \mathbb{R}^n$ has countably many diameters if the width function ω_K reaches its maximum on a countable subset of S^{n-1}. Also, a body has *constant width* if its width function is constant.

A set $E \subset \mathbb{R}^n$ is said to be *star-shaped at a point* p if the line segment from p to any point in E is contained in E. Let $x \in \mathbb{R}^n \setminus \{0\}$, and let $K \subset \mathbb{R}^n$ be a star-shaped set at the origin. The *radial function* ρ_K is defined as $\rho_K(x) = \max\{c \mid cx \in K\}$ (here, the line through x and the origin is assumed to meet K, [3, p. 18]). We say that a body K is a *star body* if K is star-shaped at the origin and its radial function ρ_K is continuous. The radial function uniquely determines a star body, and $\rho_{K_1} \le \rho_{K_2}$ if and only if $K_1 \subseteq K_2$.

For a subset E of $\mathbb{R}^n$, the *polar set* of E is defined as

$$E^* = \{x \mid x \cdot y \le 1 \quad \text{for every } y \in E\}$$

(see [3, pp. 20–22]). When K is a convex body containing the origin, the same is true of K^* (which is called the *polar body* of K), and we have the following relation between the support function of K and the radial function of K^*: for every $u \in S^{n-1}$,

$$\rho_{K^*}(u) = 1/h_K(u). \tag{2.1}$$

For any linear transformation $\varphi \in GL(n)$, we have

$$h_{\varphi K}(u) = h_K(\varphi^t u). \tag{2.2}$$

A similar relation

$$\rho_{\varphi K}(u) = \rho_K(\varphi^{-1} u) \tag{2.3}$$

holds for the radial function. By combining (2.1), (2.2), and (2.3), it follows that $h_{(\varphi K)^*}(u) = h_{\varphi^{-t}K^*}(u)$ (see [3, page 21]); this gives us the identity $(\varphi K)^* = \varphi^{-t}K^*$ for the polar of a linear transformation of the body K.

If S is a subspace of $\mathbb{R}^n$, then $h_{K|S}(u) = h_K(u)$ and $\rho_{K\cap S}(u) = \rho_K(u)$ for every $u \in S^{n-1} \cap S$. Combining these facts with (2.1), we obtain the polarity relation between sections and projections:

$$\rho_{K^* \cap S}(u) = \rho_{(K|S)^*}(u), \quad \text{for } u \in S^{n-1} \cap S. \tag{2.4}$$

3. COUNTEREXAMPLES FOR PROBLEMS 1.1 (a) and 1.3 (a)

3.1. Counterexample in $\mathbb{R}^3$. Our first counterexample is three dimensional, and it is provided by two centrally symmetric convex bodies, a cylinder C and a double cone K. We show that all *sections* of C can be rotated to fit into the corresponding sections of K, and that C cannot be rotated to fit inside K. Because of the relations (2.1), (2.3) and polarity (2.4), this will imply that all projections of K^* fit into the corresponding projections of C^* after a rotation, while no rotation of K^* is included in C^*. Thus, our two bodies provide at the same time counterexamples for 1.1 (a) and 1.3 (a).

Let $C \subset \mathbb{R}^3$ be the cylinder around the z-axis, centered at the origin, with radius r and height $2r$, where $\frac{1}{2} < r < \sqrt{2-\sqrt{3}} = 0.5176\ldots$. Let K be the double cone obtained by rotating the triangle with vertices $(0,0,\pm 1)$ and $(1,0,0)$ around the z-axis. Since $r > \frac{1}{2}$, the cylinder C is not contained in the double cone K; in fact, the condition $r > \frac{1}{2}$ is enough to guarantee that no rotation of C is contained in K (the proof of this fact is given in Appendix A, Lemma A.1).

Observe that the polar body of C is the double cone obtained by rotating the triangle with vertices $(0, 0, \pm 1/r)$ and $(1/r, 0, 0)$ around the z-axis. The polar body of K is a cylinder with radius 1 and height 2. Hence, the dilation of C^* by a factor r is equal to K, and similarly the dilation of K^* by r is equal to C. By proving that all sections of C can be rotated to fit into the sections of K, we are in fact proving that all projections of C are included in the corresponding projections of K after a rotation. Here, we present a sketch of the argument, with the detailed calculations shown in Appendix A.

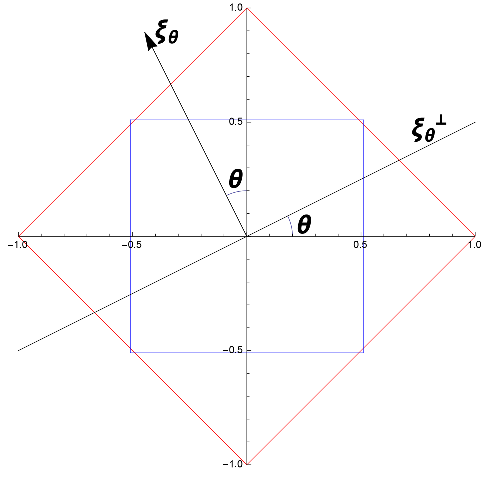

FIGURE 3.1. Cross-section of the cylinder C and the double cone K

It is enough to study their sections by planes perpendicular to

$$\xi_\theta = (-\sin(\theta), 0, \cos(\theta)),$$

where $\theta \in [0, \pi/2]$ is the vertical angle from the axis of revolution (see Figure 3.1), since C and K are centrally symmetric bodies of revolution. As seen in Appendix A, the radial function of the section of the double cone K by $\xi_\theta^\perp$ is

$$\rho_{K_\theta}(u) = \frac{\sec(u)}{\sin(\theta) + \sqrt{\tan^2(u) + \cos^2(\theta)}}.$$

When $\theta \in [0, \pi/4]$, the section of the cylinder by $\xi_\theta^\perp$ is an ellipse with semiaxes of length $r \sec\theta$ and r. Its radial function is

$$\rho_{C_\theta}(u) = \frac{r \sec(u)}{\sqrt{\tan^2(u) + \cos^2(\theta)}}. \tag{3.1}$$

On the other hand, when $\theta \in (\pi/4, \pi/2]$, the section of the cylinder looks like an ellipse with semiaxes of length $r \sec\theta$ (along the x-axis) and r (along the y-axis), which has been truncated by two vertical lines at $x = \pm r\csc(\theta)$. Its radial function is

$$\rho_{C_\theta}(u) = \begin{cases} r\sec(u)\csc(\theta) & 0 \le u \le u_0, \\ \dfrac{r\sec(u)}{\sqrt{\tan^2(u) + \cos^2(\theta)}} & u_0 \le u \le \dfrac{\pi}{2}, \end{cases}$$

where $u_0 = \arctan\sqrt{\sin^2(\theta) - \cos^2(\theta)}$.

Let $\theta_0 = \arctan((1-r)/r)$. For $\theta \in [0, \theta_0]$, the section $C \cap \xi_\theta^\perp$ is contained in $K \cap \xi_\theta^\perp$, and there is nothing to prove (see Figure 3.1). For $\theta \in (\theta_0, \pi/4]$, $C \cap \xi_\theta^\perp$ is not a subset of $K \cap \xi_\theta^\perp$. However, a rotation by $\pi/2$ of the section of the cylinder is contained in the section of the double cone. Indeed, from equation (3.1) we easily see that the rotation by $\pi/2$ of the section of the cylinder has radial function

$$\tilde{\rho}_{C_\theta}(u) = \frac{r\csc(u)}{\sqrt{\cot^2(u) + \cos^2(\theta)}}. \tag{3.2}$$

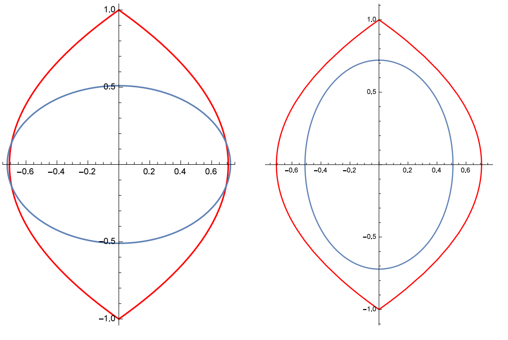

FIGURE 3.2. Left: For $\theta \in (\theta_0, \pi/4]$, the section of the cylinder is not a subset of the section of the double cone. Right: The section of the cylinder has been rotated by $\pi/2$. Here, $r = 0.51$, $\theta = \pi/4$.

We have that $\tilde{\rho}_{C_\theta}(u) < \rho_{K_\theta}(u)$, for every $u \in [0, \pi/2]$, $\theta \in (\theta_0, \pi/4]$. The crucial observation is that for a fixed u, $\tilde{\rho}_{C_\theta}(u)$ is an increasing function of $\theta \in [0, \pi/4]$, while $\rho_{K_\theta}(u)$ is decreasing. It is enough, therefore, to show that $\tilde{\rho}_{C_{\pi/4}}(u) < \rho_{K_{\pi/4}}(u)$ for $u \in [0, \pi/2]$. Figure 3.2 shows this situation for $r = 0.51$.

When $\theta \in (\pi/4, \pi/2]$, the section $C \cap \xi_\theta^\perp$ is never contained in $K \cap \xi_\theta^\perp$, but if $r < \sqrt{2 - \sqrt{3}}$, there exist angles $\theta_1, \theta_2 \in (\pi/4, \pi/2)$, with $\theta_2 \le \theta_1$, such that for $\theta \in (\pi/4, \theta_1]$, a rotation by $\pi/2$ of the section of the cylinder is contained

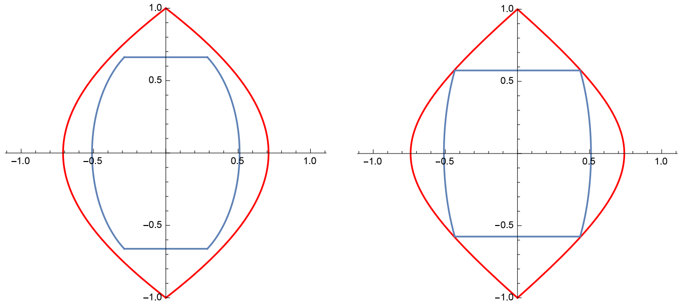

FIGURE 3.3. Section of the double cone and the $\pi/2$ rotation of the section of the cylinder for $r = 0.51$. For the left figure, $\theta \in (\pi/4, \theta_1)$; for the right figure, $\theta = \theta_1$.

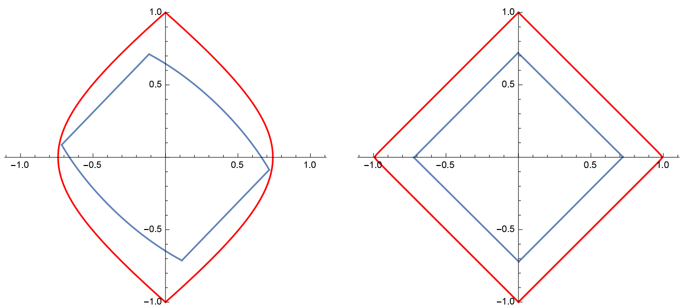

FIGURE 3.4. In both figures, $r = 0.51$. The left figure shows the same sections as Figure 3.3 (right), but the section of the cylinder has been rotated by $\pi/4$. The right figure shows the case where $\theta = \pi/2$.

in the section of the double cone, and for $\theta \in [\theta_2, \pi/2]$, a rotation by the angle u_0 of the section of the cylinder is contained in the section of the double cone. The idea is that when $\theta = \pi/4$, the $\pi/2$ rotation of the section $C \cap \xi_{\pi/4}^{\perp}$ is strictly contained within $K \cap \xi_{\pi/4}^{\perp}$, which implies the same, by continuity, for θ in some interval $(\pi/4, \theta_1]$; on the other hand, when $\theta = \pi/2$ and both sections are squares, a rotation by $\pi/4 = u_0(\pi/2)$ of $C \cap \xi_{\pi/4}^{\perp}$ is strictly included in $K \cap \xi_{\pi/4}^{\perp}$, and by continuity the same is true on some interval $[\theta_2, \pi/2]$. The calculations in Appendix A show that for $r \in (\frac{1}{2}, \sqrt{2-\sqrt{3}})$, $\theta_2 \leq \theta_1$, and hence all sections of C can be rotated to fit within the corresponding sections of K. Figures 3.3 and 3.4 illustrate both cases.

3.2. Counterexample in $\mathbb{R}^n$**.** Given the unit sphere in $\mathbb{R}^n$, we will perturb it by adding bump functions to create two convex bodies K and L. We place the bumps on K so that they form a simplex on the surface of K, but no such simplex configuration of bumps will appear on the surface on L, thus impeding K from being rotated to fit inside of L. On the other hand, every hyperplane section of K will be contained in the corresponding section of L after a rotation. We thus obtain a counterexample for 1.3 (a). By polarity, the bodies L^* and K^* provide a counterexample for Problem 1.1 (a).

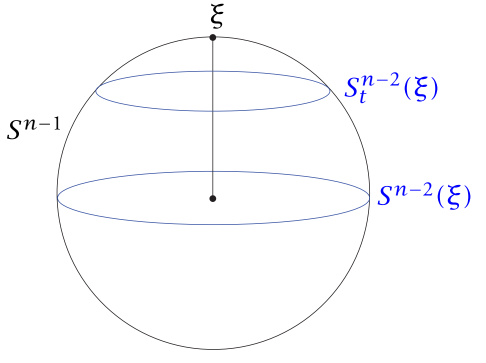

FIGURE 3.5.

Given $\xi \in S^{n-1}$, the great $(n-2)$-dimensional subsphere of S^{n-1} that is orthogonal to ξ will be denoted by $S^{n-2}(\xi) = \{\theta \in S^{n-1} \mid \theta \cdot \xi = 0\}$. For $t \in [-1, 1]$, the subsphere that is parallel to $S^{n-2}(\xi)$ and is located at height t will be denoted by $S_t^{n-2}(\xi)$ (see Figure 3.5).

The radial function of the unit sphere is the constant function 1. We consider a smooth bump function $\varphi_{\xi,\delta}$ defined on S^{n-1}, supported in a small disk D_ξ on the surface of S^{n-1} with center at $\xi \in S^{n-1}$ and with radius δ. The function $\varphi_{\xi,\delta}$ is invariant under rotations that fix the direction ξ, and its maximum height at the point ξ is 1. For $u \in S^{n-1}$, the body whose radial function is $1 + \varepsilon\varphi_{\xi,\delta}(u)$ is convex, since its curvature will be positive provided that ε is small enough (see, e.g., [4, page 267]).

The first body K is defined to be the unit sphere with n bumps placed on the surface, so that their centers ξ_j, $j = 1, \dots, n$ form a regular spherical simplex. We assume that the vertex ξ_1 is the north pole, and that $v < 4^{-n}/10^3$ is the spherical distance between the vertices of the simplex. The radial function of K is

$$1 + \sum_{j=1}^{n} \frac{\varepsilon}{10^3} \varphi_{\xi_j,\delta}(u), \quad \text{for } u \in S^{n-1};$$

that is, each bump is supported on a disk with center ξ_j and radius δ (to be chosen later), and has height $\varepsilon/10^3$, where ε is small enough so that K will be convex. Given any two vertices of the simplex ξ_i and ξ_j, with $i \neq j$, consider the lune formed by the union of all $(n-2)$-dimensional great spheres passing through any two points $x \in D_{\xi_i}$ and $y \in D_{\xi_j}$. Let a be the maximum width of the lune. If we choose $\delta = v^4$, it follows that $a \approx v^3$ (see Figure 3.6).

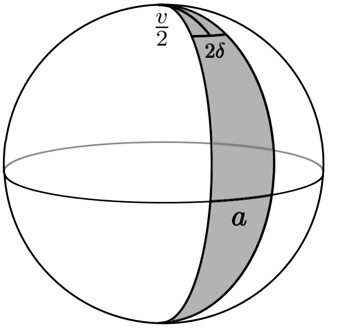

FIGURE 3.6.

We define L to be the unit sphere with bumps placed on the surface in the following way. On every point $\xi_j \neq \xi_1$, (i.e., not on the north pole), we place a bump function of height $\varepsilon/10^3$ and radius δ. Notice that these are the values of the height and radius of the bumps on K. Thus, by construction, any section of K that does not pass through the bump at the north pole is automatically contained in the corresponding section of L.

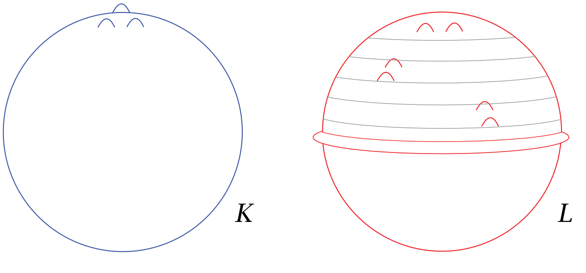

FIGURE 3.7. Placement of the bumps on K and L for $n = 3$

We now consider a section of K that passes through the bump at the north pole. We will need to place bumps on L in such a way that the section of K can be rotated to be included in the corresponding section of L. For this purpose, we split the top half of the sphere into 2^n layers. For $k = 1, \dots, 2^n - 1$, the k-th layer L_k is the spherical ring placed between the parallels $S^{n-2}_{t_{k-1}}(\xi_1)$ and $S^{n-2}_{t_k}(\xi_1)$, where $t_k = k/2^n$. The top layer is the spherical cap centered at the north pole, and above the parallel $S^{n-2}_{t_{2^n-1}}(\xi_1)$. Observe that the $(n-1)$ bumps we have already placed are all on the top layer, since $\delta < v$ and $v < 4^{-n}/10^3$, while the spherical radius of the top layer is $\frac{1}{2}\arccos(1 - 1/2^n) \approx 2/\sqrt{2^{n+1}}$. For every odd k, the layer L_k will remain empty of bumps. For each $2 \le j \le n-1$, and for each configuration of j vertices of the simplex in K, one of which is the north pole, we place on a layer L_k with k even an identical configuration of vertices (i.e., a rotation of the original configuration into L_k; see Figure 3.7). On each vertex x we place the bump function $\varepsilon\varphi_{x,\tilde{\delta}}$, where $\tilde{\delta} = v^2$. The definition of v guarantees that the layers are wide enough to contain each configuration of bumps, and also that the larger bumps do not overlap, since $2\tilde{\delta} < v$.

Since $\tilde{\delta} \gg a$, every section of K that intersects j of the bumps will be contained after a rotation in the corresponding section of L. On the other hand, since $\tilde{\delta} \ll v$, no layer can contain n bumps of smaller height $\varepsilon/10^3$ placed in the shape of the original simplex on K.

Finally, we define a function ψ_{ξ_1} as the function obtained by sliding $(\varepsilon/10^3)\varphi$ around the equator $S^{n-2}(\xi_1)$ of L. This guarantees that every section of K that passes through the north pole and no other bump on K can be rotated by the angle $\pi/2$ into the corresponding section of L. This concludes the n-dimensional counterexample.

4. SECTIONS, PROJECTIONS, AND VOLUMES

The counterexamples to Problem 1.1 (a) presented in Section 3 do not provide a negative answer to Problem 1.1 (b), as in both cases the body with the larger projections also has the larger volume. In fact, Theorem 4.2 below shows that our assumptions on the projections of K and L only imply that the volume of the polar body K^* is larger than the volume of L^*, but gives us no relation between the volumes of K and L. On the other hand, the answer to Problem 1.3 (b) is affirmative: one can obtain the desired relation $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$ if the *sections* of K are assumed to fit into the corresponding sections of L after rotation. This is proved in the following theorem.

Theorem 4.1. *Let K and L be two star bodies in $\mathbb{R}^n$, $n \ge 2$, such that for every $\xi \in S^{n-1}$, there exists a rotation $\varphi_\xi \in \mathrm{SO}(n-1, \xi^\perp)$ such that*

$$\varphi_\xi(K \cap \xi^\perp) \subseteq L \cap \xi^\perp.$$

Then, $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$.

Proof. By hypothesis, there exists a rotation $\varphi_\xi \in \mathrm{SO}(n-1, \xi^\perp)$ for every $\xi \in S^{n-1}$ such that $\rho_{\varphi_\xi(K\cap\xi^\perp)}(\theta) \le \rho_{L\cap\xi^\perp}(\theta)$ for all $\theta \in \xi^\perp$. By (2.3), this is equivalent to $\rho_K(\varphi^t_\xi(\theta)) \le \rho_L(\theta)$for all $\theta \in \xi^\perp$. Raising to the power n, integrating, and using the rotation invariance of the Lebesgue measure, we obtain

$$\int_{\xi^\perp\cap S^{n-1}} \rho_K^n(\varphi_\xi^t(\theta))\,\mathrm{d}\theta = \int_{\xi^\perp\cap S^{n-1}} \rho_K^n(\theta)\,\mathrm{d}\theta \le \int_{\xi^\perp\cap S^{n-1}} \rho_L^n(\theta)\,\mathrm{d}\theta.$$

Averaging over the unit sphere, we have

$$\int_{S^{n-1}} \mathrm{d}\xi \int_{\xi^\perp\cap S^{n-1}} \rho_K^n(\theta)\,\mathrm{d}\theta \le \int_{S^{n-1}} \mathrm{d}\xi \int_{\xi^\perp\cap S^{n-1}} \rho_L^n(\theta)\,\mathrm{d}\theta.$$

Finally, using Fubini's theorem and the formula for the volume in terms of the radial function (see [8, page 16]), namely,

$$\mathrm{vol}_n(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(\theta)\,\mathrm{d}\theta, \tag{4.1}$$

we obtain the result. □

For the next theorem, we use the standard notation $\mathrm{int}(K)$ for the interior of K. The proof is similar to that of Theorem 4.1.

Theorem 4.2. *Let K and L be two convex bodies in $\mathbb{R}^n$, with $n \ge 2$, such that $0 \in \mathrm{int}(K) \cap \mathrm{int}(L)$, and such that, for every $\xi \in S^{n-1}$, there exists a rotation $\varphi_\xi \in \mathrm{SO}(n-1,\xi^\perp)$ such that $\varphi_\xi(K|\xi^\perp) \subseteq L|\xi^\perp$. Then,*

$$\mathrm{vol}_n(K^*) \ge \mathrm{vol}_n(L^*).$$

Proof. By hypothesis, there exists a rotation $\varphi_\xi \in \mathrm{SO}(n-1,\xi^\perp)$ for every $\xi \in S^{n-1}$ such that $h_{\varphi_\xi(K|\xi^\perp)}(\theta) \le h_{L|\xi^\perp}(\theta)$ for all $\theta \in \xi^\perp$. By (2.1) and (2.2), this is equivalent to $\rho_{K^*}(\varphi_\xi^t(\theta)) \ge \rho_{L^*}(\theta)$for all $\theta \in \xi^\perp$. Raising to the power n, integrating, and using the rotation invariance of the Lebesgue measure, we obtain

$$\int_{\xi^\perp\cap S^{n-1}} \rho_{K^*}^n(\varphi_\xi^t(\theta))\,\mathrm{d}\theta = \int_{\xi^\perp\cap S^{n-1}} \rho_{K^*}^n(\theta)\,\mathrm{d}\theta \ge \int_{\xi^\perp\cap S^{n-1}} \rho_{L^*}^n(\theta)\,\mathrm{d}\theta.$$

Averaging over the unit sphere we have

$$\int_{S^{n-1}} \mathrm{d}\xi \int_{\xi^\perp\cap S^{n-1}} \rho_{K^*}^n(\theta)\,\mathrm{d}\theta \ge \int_{S^{n-1}} \mathrm{d}\xi \int_{\xi^\perp\cap S^{n-1}} \rho_{L^*}^n(\theta)\,\mathrm{d}\theta.$$

Finally, using Fubini's Theorem and (4.1), we obtain the desired result. □

To obtain a positive answer to Problem 1.1 (b), we need to impose additional conditions on the bodies K and L. We do this in Theorem 4.3, following ideas of Hadwiger [6], by assuming the existence of a diameter $d_K(\xi_0)$ of K in the direction ξ_0, such that the hypotheses of Problem 1.1 hold on every plane that contains that diameter. For $w \in \xi_0^\perp$, we will call $K|w^\perp$ (respectively, $L|w^\perp$) a *side projection* of K (respectively of L); see Figure 4.1.

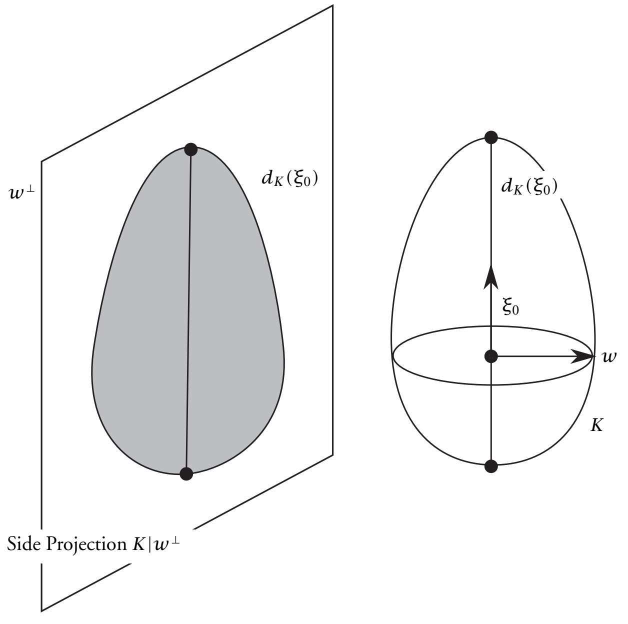

FIGURE 4.1.

Theorem 4.3. *Let K, L be convex bodies in $\mathbb{R}^3$ such that each of K, L have countably many diameters of the same length, that is,*

$$\max_{\{\theta \in S^{n-1}\}} \omega_K(\theta) = \max_{\{\theta \in S^{n-1}\}} \omega_L(\theta),$$

and, for each body, the maximum is attained at a countable set of directions. Assume that there exists a diameter $d_K(\xi_0)$, such that for every $w \in \xi_0^\perp$, there exists $\varphi_w \in \mathrm{SO}(2, w^\perp)$ such that $\varphi_w(K|w^\perp) \subseteq L|w^\perp$; that is, every side projection of K is contained, after a rotation, in the corresponding side projection of L. If either K or L is origin-symmetric, then $K \subseteq L$.

Proof. First, we show that L must have a diameter in the direction ξ_0. If this is not the case, then there exists a plane H that contains ξ_0 and none of the directions of the diameters of L. Then, $K|H$ contains a diameter of K (namely, $d_K(\xi_0)$) and $L|H$ does not contain a diameter of L; hence, $K|H$ can never be rotated into $L|H$. This contradiction shows that L has a diameter $d_L(\xi_0)$.

Let D be the countable set of all directions of the diameters of K and L, excluding ξ_0. For $w \in \xi_0^\perp$, let $w^\perp$ be a plane containing no direction in D (clearly, $w^\perp$ contains ξ_0). Since we have that $\varphi_w(K|w^\perp) \subseteq L|w^\perp$, and φ_w is a rotation around the origin that takes the diameter $d_K(\xi_0)$ onto the parallel diameter $d_L(\xi_0)$, it follows that either φ_w is the identity, or that both $d_K(\xi_0)$ and $d_L(\xi_0)$ are centered at the origin and φ_w is a rotation by π. In the first case, we have $K|w^\perp \subseteq L|w^\perp$. In the second case, we have $-K|w^\perp \subseteq L|w^\perp$. But either K or L is origin symmetric, and hence we obtain $K|w^\perp \subseteq L|w^\perp$ also in this case. Thus, for every $\theta \in S^2$ such that $\theta \in w^\perp$ and $w^\perp$ does not contain any direction in D, we have that $h_K(\theta) \leq h_L(\theta)$.

Let H_i be the plane containing ξ_0 and $\xi_i \in D$, and assume that $\theta \in S^2 \cap H_i$. Since there are only countably many such H_i, we can choose a sequence $\{\theta_j\}$ of points in S^2, converging to θ, such that none of the θ_j are contained in $\bigcup_{i\geq 1} H_i$. Hence, $h_K(\theta_j) \leq h_L(\theta_j)$, and by the continuity of the support function, $h_K(\theta) \leq h_L(\theta)$. We have thus proven that $K \subseteq L$. □

Remark 4.4. A more general version of Theorem 4.3 can be proven: With the same hypotheses for the diameters of K, L, if every projection $K|w^\perp$ can be rotated *and translated* to be included in $L|w^\perp$, and either K or L is centrally symmetric, then K is contained in a translate of L (the argument is similar to that in the proof of Lemma 14 in [1]).

We finish this section with two related results.

Lemma 4.5. *Let K, L be two convex bodies in $\mathbb{R}^3$ such that*

$$\forall\, \xi \in S^2,\ \exists\, \varphi_\xi \in \mathrm{SO}(2, \xi^\perp) \quad \text{such that } \varphi_\xi(K|\xi^\perp) \subseteq L|\xi^\perp$$

and

$$\int_{S^2} h_K = \int_{S^2} h_L.$$

Then, $K \subseteq \pm L$.

Proof. Assume $\varphi_\xi(K|\xi^\perp)$ is strictly contained in $L|\xi^\perp$. By continuity, there is an open set of directions in S^2 where the containment is strict. Integrating, we obtain $\int_{S^2} h_K < \int_{S^2} h_L$, contradicting our hypothesis. Therefore, for every $\xi \in S^2$ there exists a rotation $\varphi_\xi \in \mathrm{SO}(2, \xi^\perp)$ such that $\varphi_\xi(K|\xi^\perp) = L|\xi^\perp$. By [10], we conclude that $K = \pm L$. □

Lemma 4.6. *Let K, L be two convex bodies of equal constant width in $\mathbb{R}^3$, such that*

$$\forall\, \xi \in S^2,\ \exists\, \varphi_\xi \in \mathrm{SO}(2, \xi^\perp) \quad \text{such that } \varphi_\xi(K|\xi^\perp) \subseteq L|\xi^\perp.$$

Then, $\mathrm{vol}_3(K) \leq \mathrm{vol}_3(L)$.

Proof. The assumption on the projections implies that the surface area of K is less than or equal to the surface area of L. Indeed, Cauchy's surface area formula [3, page 408] says that the surface area of the body K is equal to

$$S(K) = \frac{1}{\mathrm{vol}_{n-1}(B)} \int_{S^{n-1}} \mathrm{vol}_{n-1}(K|u^\perp)\, \mathrm{d}u,$$

where B is the unit Euclidean ball in $\mathbb{R}^n$. Since every projection of K is contained in the corresponding projection of L after a rotation, we have $\mathrm{vol}_2(K|u^\perp) \le \mathrm{vol}_2(L|u^\perp)$, and Cauchy's formula gives us $S(K) \le S(L)$.

On the other hand, for bodies of constant width w in $\mathbb{R}^3$, there is a known formula relating volume, surface area, and width (see [2, page 66]):

$$2\,\mathrm{vol}_3(K) = wS(K) - \frac{2\pi}{3}w^3.$$

Using this relation, we conclude that $\mathrm{vol}_3(K) \le \mathrm{vol}_3(L)$. □

5. Directly Congruent Projections

In the previous sections, we have studied the problem in which the projections of K can be rotated to be contained into the corresponding projections of L. In this section, we assume that the projections of K are directly congruent to the corresponding projections of L (i.e., they coincide up to a rotation and a translation). The main argument of the proof follows ideas from Golubyatnikov [5] and Ryabogin [10]. Some of the arguments we use can be found in full detail in the paper [1].

Theorem 5.1. *Let K and L be two convex bodies in $\mathbb{R}^3$ having countably many diameters. Assume there exists a diameter $d_K(\xi_0)$, such that the side projections $K|w^\perp$, $L|w^\perp$ onto all subspaces $w^\perp$ containing ξ_0 are directly congruent. Assume also that these projections are not centrally symmetric. Then, $K = \pm L + b$ for some $b \in \mathbb{R}^3$.*

Proof. For the same reasons as in the proof of Theorem 4.3, L must have a diameter parallel to the direction ξ_0. Denote by $\{\xi_0, \xi_1, \dots\}$ the countable set of directions parallel to the diameters of K and the diameters of L. Let H_i be the plane that contains the directions ξ_0 and ξ_i, and consider the set

$$\Lambda = \Big\{w \in S^1(\xi_0) \mid w \notin \bigcup_i H_i\Big\}.$$

Because the set $\{H_i\}$ is countable, Λ is everywhere dense in $S^1(\xi_0)$. (In fact, the hypothesis that K and L have countably many diameters can be replaced by the weaker hypothesis that Λ is everywhere dense.)

Next, we translate K and L so that their diameters $d_K(\xi_0)$ and $d_L(\xi_0)$ coincide and are centered at the origin. We name the translated bodies $\tilde{K}$ and $\tilde{L}$. It is easy to show that the side projections $\tilde{K}|w^\perp$ and $\tilde{L}|w^\perp$ coincide up to a rotation (see [1, Lemma 14]).

Denote by H'_w the plane that contains w and ξ_0. Since for all $w \in \Lambda$ the only diameter of $\tilde{K}|H'_w$ is $d_{\tilde{K}}(\xi_0)$, the direct rigid motion given by the statement of the theorem must fix this diameter, and hence it is either the identity or a rotation about the origin by π. In the first case, $\tilde{K}|H'_w = \tilde{L}|H'_w$, and in the second,

$\tilde{K}|H'_w = -\tilde{L}|H'_w$. Define

$$\Xi = \{w \in S^1(\xi_0) : \tilde{K}|H'_w = \tilde{L}|H'_w\}$$

and

$$\Psi = \{w \in S^1(\xi_0) : \tilde{K}|H'_w = -\tilde{L}|H'_w\}.$$

By a standard argument, it is easy to prove that Ξ and Ψ are closed (see Lemma 3 in [1]). From the definitions of Ξ and Ψ we have that $\Lambda \subseteq \Xi \cup \Psi \subseteq S^1(\xi_0)$, and since $\Xi \cup \Psi$ is closed and Λ is everywhere dense, we have $\Xi \cup \Psi = S^1(\xi_0)$.

Next, we claim that $\Xi \cap \Psi = \emptyset$. Indeed, assume this is not the case, and let $w \in \Xi \cap \Psi$. Then, we have $\tilde{L}|H'_w = \tilde{K}|H'_w = -\tilde{L}|H'_w$, which implies that $\tilde{L}|H'_w$ (and hence $L|H'_w$) is centrally symmetric. This contradicts our assumption, and thus $\Xi \cap \Psi = \emptyset$.

Therefore, either $\Xi = S^1(\xi_0)$ or $\Psi = S^1(\xi_0)$. If $\Xi = S^1(\xi_0)$, then for every $\theta \in S^2$ there exists a plane H'_w for some $w \in S^1(\xi_0)$, such that $\theta \in H'_w$. Hence,

$$h_{\tilde{K}}(\theta) = h_{\tilde{K}|H'_w}(\theta) = h_{\tilde{L}|H'_w}(\theta) = h_{\tilde{L}}(\theta),$$

where the second equality follows from the definition of Ξ. It follows that $\tilde{K} = \tilde{L}$.

If $\Psi = S^1(\xi_0)$, again for every $\theta \in S^2$, we have that $\theta \in H'_w$ for some $w \in S^1(\xi_0)$. Note that we also have $-\theta \in H'_w$. Hence,

$$h_{\tilde{K}}(\theta) = h_{\tilde{K}|H'_w}(\theta) = h_{-\tilde{L}|H'_w}(\theta) = h_{(-\tilde{L})|H'_w}(\theta) = h_{-\tilde{L}}(\theta),$$

where the second equality follows from the definition of Ψ. We conclude that $\tilde{K} = -\tilde{L}$, and the theorem is proved. □

Appendix A. The Sections of the Cylinder and the Double Cone in $\mathbb{R}^3$

Here, for the convenience of the reader we provide the detailed calculations for the example in Section 3.

Determining the radial function of the boundary curves of the sections of K and C. The upper half of the double cone has equation $z = 1 - \sqrt{x^2 + y^2}$, and the plane $\xi_\theta^\perp$ has equation $z = \tan(\theta)x$. The curve of intersection in parametric equations is given by

$$r_{K,\theta}(t) = \langle (1-z)\cos(t), (1-z)\sin(t), z \rangle,$$

where $z = \tan(\theta)(1-z)\cos(t)$ (from the equation of the plane). Solving for z in this last equation, we obtain

$$z = \frac{\tan(\theta)\cos(t)}{1 + \tan(\theta)\cos(t)},$$

and therefore

$$r_{K,\theta}(t) = \left\langle \frac{\cos(t)}{1+\tan(\theta)\cos(t)}, \frac{\sin(t)}{1+\tan(\theta)\cos(t)}, \frac{\tan(\theta)\cos(t)}{1+\tan(\theta)\cos(t)} \right\rangle.$$

This curve is still expressed as a subset of $\mathbb{R}^3$, so now we will write it as a two dimensional curve in the plane $\xi_\theta^\perp$. The vectors $\langle 1,0,0\rangle$ and $\langle 0,1,0\rangle$ project onto vectors with the same direction as

$$\vec{e}_{1,\theta} = \left\langle \frac{1}{\sqrt{1+\tan^2(\theta)}}, 0, \frac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}} \right\rangle = \langle \cos(\theta), 0, \sin(\theta)\rangle$$

and $\vec{e}_{2,\theta} = \langle 0,1,0\rangle$ in the plane $z = \tan(\theta)x$. Consequently, for $t \in [0,\pi/2]$, the parametric curve written in this basis becomes

$$\tilde{r}_{K,\theta}(t) = \left(\frac{\cos(t)\sec(\theta)}{1+\tan(\theta)\cos(t)}\right)\vec{e}_{1,\theta} + \left(\frac{\sin(t)}{1+\tan(\theta)\cos(t)}\right)\vec{e}_{2,\theta}.$$

Finally, it will be more convenient to express this in polar coordinates. Setting $\tilde{r}_{K,\theta}(t) = \rho_{K_\theta}(u)\cos(u)\vec{e}_{1,\theta} + \rho_{K_\theta}(u)\sin(u)\vec{e}_{2,\theta}$ and solving, we obtain that the radial function of the section $K \cap \xi_\theta^\perp$ is

$$\rho_{K_\theta}(u) = \frac{\sec(u)}{\sin(\theta) + \sqrt{\tan^2(u)+\cos^2(\theta)}}, \quad \text{for } u \in [0,\pi/2].$$

The function is extended evenly to $[-\pi/2, 0]$. It can easily be checked that $\rho'_{K_\theta}(u) \geq 0$ when $\theta \in [0,\pi/4]$, and thus $\rho_{K_\theta}(u)$ is an increasing function of u on $[0,\pi/2]$, with maximum value $\rho_{K_\theta}(\pi/2) = 1$, and minimum value $\rho_{K_\theta}(0) = 1/(\sin\theta + \cos\theta)$. Also, for fixed $u \in [0,\pi/2]$, we have that ρ_{K_θ} is a decreasing function of $\theta \in [0,\pi/4]$. In contrast, when $\theta \in (\pi/4, \pi/2]$, $\rho_{K_\theta}(u)$ has a local maximum at $u = 0$ and a local (and absolute) minimum at $u_0 = \arctan\sqrt{\sin^2(t) - \cos^2(t)}$, with value $\rho_{K_\theta}(u_0) = 1/\sqrt{2}$. Its absolute maximum is $\rho_{K_\theta}(\pi/2) = 1$.

Similarly, we calculate the radial function of $C \cap \xi_\theta^\perp$. The intersection of the cylinder with the plane $z = \tan(\theta)x$, for $\theta \in [0,\pi/4]$, is an ellipse with parametrization

$$r_{C,\theta}(t) = \langle r\cos(t), r\sin(t), r\cos(t)\tan(\theta)\rangle.$$

In terms of the basis $\{\vec{e}_{1,\theta}, \vec{e}_{2,\theta}\}$, the parametrization is given by

$$\tilde{r}_{C,\theta}(t) = r\cos(t)\sec(\theta)\vec{e}_{1,\theta} + r\sin(t)\vec{e}_{2,\theta},$$

and the radial function is

$$\rho_{C_\theta}(u) = \frac{r\sec(u)}{\sqrt{\tan^2(u)+\cos^2(\theta)}},$$

and evenly extended on $[-\pi/2, 0]$. The section is an ellipse with semiaxes of length $r \sec\theta$ (for $u = 0$) and r (for $u = \pi/2$), and the radial function is strictly decreasing on $u \in [0, \pi/2]$. Here, it is also useful to note that, for fixed u, ρ_{C_θ} is an increasing function of $\theta \in [0, \pi/4]$.

When $\theta \in [\pi/4, \pi/2]$, the plane cuts the top and bottom of the cylinder, and we obtain the following radial function:

$$\rho_{C_\theta}(u) = \begin{cases} r \sec(u) \csc(\theta) & 0 \le u \le u_0, \\ \dfrac{r \sec(u)}{\sqrt{\tan^2(u) + \cos^2(\theta)}} & u_0 \le u \le \dfrac{\pi}{2}, \end{cases}$$

where $u_0 = \arctan\sqrt{\sin^2(\theta) - \cos^2(\theta)}$. The section looks like an ellipse with semiaxes of length $r \sec\theta$ (along the x-axis) and r (along the y-axis), which has been truncated by two vertical lines at $x = \pm r \csc(\theta)$. Note that the function $\rho_{C_\theta}(u)$ has a local minimum at $u = 0$, is strictly increasing on $(0, u_0)$, reaches a local (and absolute) maximum at $u = u_0$ with $\rho_{C_\theta}(u_0) = \sqrt{2}r$, and is decreasing on $(u_0, \pi/2)$. The absolute minimum is $\rho_{C_\theta}(\pi/2) = r$. Observe that the absolute maximum of ρ_{C_θ} occurs at the same point as the absolute minimum of ρ_{K_θ}, and that $\sqrt{2}r = \rho_{C_\theta}(u_0) > \rho_{K_\theta}(u_0) = 1/\sqrt{2}$, since $r > \frac{1}{2}$, thus reflecting the fact that for $\theta > \pi/4$, the section of the cylinder is not contained in the section of the cone. Figure A.1 shows the graphs of $\rho_{K_\theta}(u)$ and $\rho_{C_\theta}(u)$ with $u \in [0, \pi/2]$, for $r = 0.51$. On the left, $\theta = \pi/4$; on the right, $\pi/4 < \theta < \pi/2$.

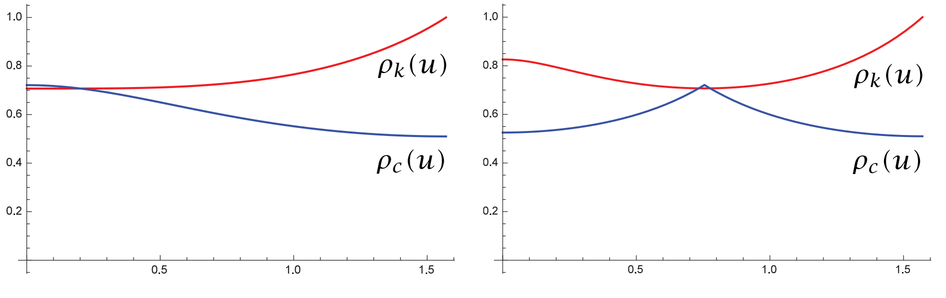

FIGURE A.1. Left: $\theta = \pi/4$; Right: $\pi/4 < \theta < \pi/2$.

We are now ready to compare the sections of the cylinder and the cone on each plane $\xi_\theta^\perp$.

As noted in Section 3, if $\theta_0 = \arctan((1-r)/r)$, for $\theta \in [0, \theta_0]$, the section of the cylinder is contained in the section of the cone, and there is nothing to prove. For $\theta \in (\theta_0, \pi/4]$, the section of the cylinder is not contained in the section of the cone, but a rotation by $\pi/2$ of the section of the cylinder is contained in the section of the cone. Since, for fixed u, $\tilde{\rho}_{C_\theta}(u)$ is increasing as a function of $\theta \in (\theta_0, \pi/4]$, while $\rho_{K_\theta}(u)$ is decreasing, it is enough for us to show that for

$u \in [0, \pi/2]$, $\tilde{\rho}_{C_{\pi/4}}(u) < \rho_{K_{\pi/4}}(u)$. Here, $\tilde{\rho}_{C_{\pi/4}}(u)$ is the radial function of the rotation by $\pi/2$ of the section of the cone, as defined in equation (3.2). We want to show that

$$\text{(A.1)} \qquad \frac{r^2 \csc^2(u)}{1/2 + \cot^2(u)} < \frac{\sec^2(u)}{\left(1/\sqrt{2} + \sqrt{\tan^2(u) + 1/2}\right)^2},$$

for $r \in (\frac{1}{2}, \sqrt{2 - \sqrt{3}})$ and $u \in [0, \pi/2]$. This can be rearranged as

$$r^2 \left(1 + \tan^2(u) + \sqrt{2}\sqrt{\tan^2(u) + \frac{1}{2}}\right) < \tan^2(u)\left(\frac{1}{2} + \cot^2(u)\right),$$

or

$$\sqrt{2}r^2\sqrt{\tan^2(u) + \frac{1}{2}} < \tan^2(u)\left(\frac{1}{2} - r^2\right) + (1 - r^2).$$

Squaring both sides, we obtain

$$0 < \frac{1}{4}(1 - 2r^2)^2 \tan^4 u + (1 - 3r^2)\tan^2 u + (1 - 2r^2),$$

a quadratic equation in $\tan^2 u$ whose discriminant is

$$(1 - 3r^2)^2 - (1 - 2r^2)^3 = r^4(8r^2 - 3).$$

But this expression is negative for $r \in (\frac{1}{2}, \sqrt{2 - \sqrt{3}})$, and thus (A.1) holds.

Calculation of the angles θ_1, θ_2**.** As noted in Section 3, when $\theta = \pi/4$, the rotation by $\pi/2$ of $C \cap \xi_\theta^\perp$ is strictly contained in the section of the double cone, and by continuity the same is true for $\theta \in (\pi/4, \theta_1)$ for some angle θ_1. Similarly, for $\theta = \pi/2$ the rotation of the section of the cylinder by $u_0 = \pi/4$ is strictly contained in the section of the double cone, and therefore the same must hold for $\theta \in (\theta_2, \pi/2)$. Here, we compute θ_1 and θ_2, and prove that $\theta_2 < \theta_1$, allowing us to always rotate the section of the cylinder to fit into the section of the cone.

Let

$$\tilde{\rho}_{C_\theta}(u) = \begin{cases} \dfrac{r\csc(u)}{\sqrt{\cot^2(u) + \cos^2(\theta)}} & 0 \le u \le \dfrac{\pi}{2} - u_0, \\ r\csc(u)\csc(\theta) & \dfrac{\pi}{2} - u_0 \le u \le \dfrac{\pi}{2}, \end{cases}$$

be the radial function of the rotation by $\pi/2$ of the section of the cylinder for $\theta \in (\pi/4, \pi/2]$. Observing Figures 3.3 and A.2, we notice that the sections of the cone and the cylinder will touch first at the "corner" point $u = \pi/2 - u_0$, where $\tilde{\rho}_{C_\theta}(u)$ has its maximum. Thus, we will define θ_1 as the angle such that

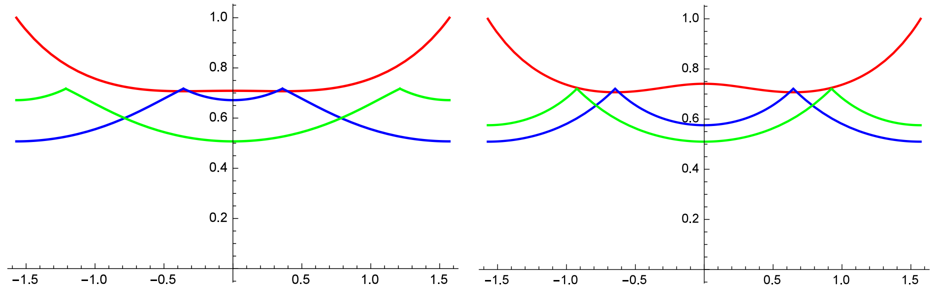

FIGURE A.2. The radial functions of the sections of the cone (red), cylinder (blue), and the rotation of the cylinder by $\pi/2$ (green). In both figures, $\theta \in (\pi/4, \theta_1)$. On the left, θ is close to $\pi/4$; on the right, θ is close to θ_1.

$\tilde{\rho}_{C_{\theta_1}}(\pi/2-u_0) = \rho_{K_{\theta_1}}(\pi/2-u_0)$. As seen above, $\tilde{\rho}_{C_{\theta_1}}(\pi/2-u_0) = \sqrt{2}r$, while for the cone we have

$$\rho_{K_\theta}\left(\frac{\pi}{2} - u_0\right) = \frac{\sqrt{2}}{(1+\sqrt{-1-2\sec(2\theta)})\sqrt{\sin^2(\theta)-\cos^2(\theta)}}.$$

These two expressions will be equal if

$$\begin{aligned} r^{-2} &= (1+\sqrt{-1-2\sec(2\theta_1)})^2(\sin^2(\theta_1)-\cos^2(\theta_1)) \\ &= 2-2\cos(2\theta_1)\sqrt{-1-2\sec(2\theta_1)}, \end{aligned}$$

or equivalently, $-4\cos(2\theta)(2+\cos(2\theta)) = (2-r^{-2})^2$; this is a quadratic equation on $\cos(2\theta)$, with solutions $-1 \pm \sqrt{1-(2-r^{-2})^2/4}$. Only the positive sign makes sense, and we obtain that the two radial functions are equal at $u = \pi/2 - u_0$ only for $\theta = \theta_1$, where

$$\theta_1 = \frac{1}{2}\arccos\left(-1 + \frac{\sqrt{4r^2-1}}{2r^2}\right).$$

We now compute θ_2. Let $\hat{\rho}_{C_\theta}(u) = \rho_{C_\theta}(u-u_0)$. By the above considerations on ρ_{C_θ}, the two absolute maxima of $\hat{\rho}_{C_\theta}$ happen at $u = 0$ and $u = 2u_0$; the local minima happen at $u = -\pi/2 + u_0$ and at $u = u_0$ (see Figure A.3). At the point $u = 0$ where $\hat{\rho}_{C_\theta}$ has a maximum with value $\sqrt{2}r$, ρ_{K_θ} has a local maximum with value $1/(\sin\theta + \cos\theta)$. The two values coincide for

$$\theta_2 = \frac{1}{2}\arcsin\left(\frac{1}{2r^2} - 1\right),$$

and $\hat{\rho}_{C_\theta}(0) < \rho_{K_\theta}(0)$ for $\theta > \frac{1}{2}\arcsin(1/(2r^2) - 1)$. We claim $\hat{\rho}_{C_\theta}(u) < \rho_{K_\theta}(u)$ for every $u \in [-\pi/2, \pi/2]$ and $\theta \in (\theta_2, \pi/2]$. In fact, the slope at $u = 0$ for ρ_K is zero, while for $\hat{\rho}'_{C_\theta}(0+)$ it is negative, so it decreases faster; both functions attain their local minimum at $u = u_0$, with $\hat{\rho}_{C_\theta}(u_0) = r\csc\theta$ and $\rho_{K_\theta}(u_0) = 1/\sqrt{2}$. But $r\csc\theta_2 < 1/\sqrt{2}$ for $r \in (\frac{1}{2}, \sqrt{2-\sqrt{3}})$, and $r\csc\theta$ is decreasing in θ. Hence, the cylinder function stays below the cone up to $u = u_0$. And at the other maximum for the cylinder, $\hat{\rho}_{C_\theta}(2u_0) < \rho_{K_\theta}(2u_0)$.

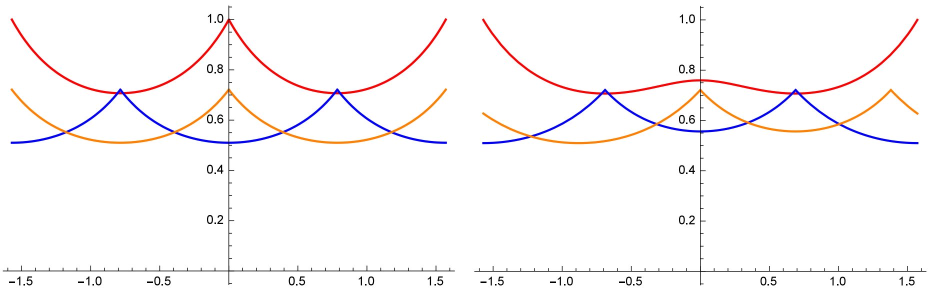

FIGURE A.3. The radial functions of the sections of the cone (red), cylinder (blue) and the rotation of the cylinder by u_0 (orange). The left figure shows the case $\theta = \pi/2$, and the right one $\theta = \theta_2$.

Finally, let us check that $\theta_2 < \theta_1$ for $r \in (\frac{1}{2}, \sqrt{2-\sqrt{3}})$. Indeed,

$$\cos(2\theta_1) = -1 + \sqrt{4r^2 - 1}/(2r^2),$$

while

$$\cos(2\theta_2) = -\sqrt{4r^2 - 1}/(2r^2),$$

and the angles will be equal if $\sqrt{4r^2-1}/r^2 = 1$, or $r^4 - 4r^2 + 1 = 0$, which has solutions $r = \pm\sqrt{2 \pm \sqrt{3}}$. Since for $r = \frac{1}{2}$, we have $\pi/4 = \theta_2 < \theta_1 = \pi/2$, the same relation holds for $r \in (\frac{1}{2}, \sqrt{2-\sqrt{3}})$. We have thus proved that all sections of the cylinder can be rotated into the corresponding section of the double cone.

Lemma A.1. *No three-dimensional rotation of cylinder C fits inside the cone K.*

Proof. By construction, $C \not\subseteq K$. Since both C and K are origin symmetric and rotationally symmetric, it is enough to consider rotations of C around the x-axis by an angle $\varphi \in (0, \pi/2]$. We will show that for each angle $\varphi \in (0, \pi/2]$, there is a point $P(\varphi)$ on the top rim of C that remains outside of K after a rotation by the angle φ around the x-axis. Consider the point

$$P(\varphi) = (r\cos\alpha_0, r\sin\alpha_0, r),$$

where $\alpha_0 = \arcsin((1-\cos\varphi)/\sin\varphi)$. The rotation of angle φ maps $P(\varphi)$ to the point

$$\begin{aligned}R(\varphi) &= (r\cos\alpha_0, r\sin\alpha_0\cos\varphi - r\sin\varphi, r\sin\alpha_0\sin\varphi + r\cos\varphi)\\ &= \left(r\frac{\sqrt{\sin^2\varphi-(1-\cos\varphi)^2}}{\sin\varphi}, r\frac{\cos\varphi-1}{\sin\varphi}, r\right).\end{aligned}$$

Note that the z-coordinate is positive, and hence it will be enough to show that $R(\varphi)$ is outside the top part of the cone K, whose equation is $z = 1-\sqrt{x^2+y^2}$. But it is clear that

$$\begin{aligned}&1-\sqrt{\left(\frac{r\sqrt{\sin^2\varphi-(1-\cos\varphi)^2}}{\sin\varphi}\right)^2+\left(\frac{r(\cos\varphi-1)}{\sin\varphi}\right)^2}\\ &\quad = 1-r < \frac{1}{2} < r.\end{aligned}$$

Therefore, $R(\varphi)$ is outside the cone, and no three-dimensional rotation of the cylinder fits inside the cone. ◻

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