A Course in Convexity, by Barvinok
Solutions to Selected Problems

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Chapter 1: Convex Sets at Large

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1. (a) Prove that the convex hull of a set is a convex set.

Proof. To show that $\operatorname{conv}(B)$ is convex, let $x, y \in B$. Then $x = \sum_{i=1}^{m} \alpha_i x_i$ and $y = \sum_{j=1}^{n} \beta_j y_j$ with $x_i, y_j \in B$, $\alpha_i, \beta_j \geq 0$ and $\sum \alpha_i = \sum \beta_j = 1$.
Now, for any $0 \leq \lambda_0 \leq 1$,

$$\lambda_0 x + (1-\lambda_0) y = \lambda_0 \left(\sum_{i=1}^{m} \alpha_i x_i \right) + (1-\lambda_0) \left(\sum_{j=1}^{n} \beta_j y_j \right)$$

$$= \left(\sum_{i=1}^{m} \lambda_0 \alpha_i x_i \right) + \left(\sum_{j=1}^{n} (1-\lambda_0) \beta_j y_j \right).$$

Let $\gamma_k = \begin{cases} \lambda_0 \alpha_k & 1 \leq k \leq m \\ (1-\lambda_0)\beta_{k-m} & m < k \leq m+n \end{cases}$ and $z_k = \begin{cases} x_k & 1 \leq k \leq m \\ y_{k-m} & m < k \leq m+n. \end{cases}$

Then, $\sum \gamma_k = \lambda_0 \sum \alpha_i + (1-\lambda_0) \sum \beta_j = 1$, so $\lambda_0 x + (1-\lambda_0) y = \sum_{k=1}^{m+n} \gamma_k z_k$ is a convex combination of points in B. Therefore $\lambda_0 x + (1-\lambda_0) y \in \operatorname{conv}(B)$. Since λ_0 was arbitrary, the segment $[x, y] \subseteq \operatorname{conv}(B)$. □

(b) Prove that $\operatorname{conv}(B)$ is the smallest convex set containing B.

Proof. Let C be a convex set containing B, such that $B \subseteq C \subseteq \operatorname{conv}(B)$. We aim to show that any convex combination of points in B is an element of C, that is, $C \supseteq \operatorname{conv}(B)$. For an arbitrary convex combination of n points of B, induct on n.
Base case: A convex combination of one point $x \in B \subseteq C$ is just x itself, and $x \in C$.
Induction step: Suppose that any convex combination of $n-1$ points of B is contained in C. Consider the convex combination $\sum_{i=1}^{n} \alpha_i x_i$, where $x_i \in B$ for all $1 \leq i \leq n$. Let $\beta = \sum_{i=1}^{n-1} \alpha_i = 1 - \alpha_n$. Then $\displaystyle\sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} x_i \in C$ by the inductive hypothesis, since $0 \leq$

$\dfrac{\alpha_i}{\beta}$ and $\displaystyle\sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} = 1$.

Since C is convex, the line segment $\left[\displaystyle\sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} x_i, x_n\right] \subseteq C$. Because $\beta \leq 1$, we have

$$\beta\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} x_i\right) + (1-\beta)x_n = \left(\sum_{i=1}^{n-1} \alpha_i x_i\right) + \alpha_n x_n = \sum_{i=1}^{n} \alpha_i x_i \in C.$$

□

2. We define a **Polyhedron** in $\mathbb{R}^d$ as the set

$$A = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{c_i} \cdot \mathbf{x} \leq \beta_i \text{ for each } i \in \{1, 2, ..., n\}\},$$

where $\mathbf{c_i} \in \mathbb{R}^d$ are constant vectors, and $\beta_i \in \mathbb{R}$ are real numbers.

We prove that any polyhedron is convex.

Proof. Let A be the polyhedron described above. and suppose $\mathbf{x}$, $\mathbf{y} \in A$. By definition, we then have $\mathbf{c_i} \cdot \mathbf{x} \leq \beta_i$ and $\mathbf{c_i} \cdot \mathbf{y} \leq \beta_i$

We prove that the segment $[\mathbf{x}, \mathbf{y}]$ is entirely contained in A. Let $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$. Then, there exists $\lambda \in [0, 1]$ such that $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$. We show $\mathbf{z} \in A$
For all $i \in \{1, ..., n\}$ we have:

$$\begin{aligned}
\mathbf{c_i} \cdot \mathbf{z} &= \mathbf{c_i} \cdot (\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \\
&= \lambda(\mathbf{c_i} \cdot \mathbf{x}) + (1-\lambda)(\mathbf{c_i} \cdot \mathbf{y}) \text{ (Bilinearity of the dot product)} \\
&\leq \lambda\beta_i + (1-\lambda)\beta_i \text{ (because } \lambda \geq 0, 1-\lambda \geq 0 \text{)} \\
&= \beta_i.
\end{aligned}$$

This shows that $\mathbf{z} \in A$. Since $\mathbf{z}$ was arbitrary, it follows that $[\mathbf{x}, \mathbf{y}] \subset A$, and since $\mathbf{x}, \mathbf{y}$ were arbitrary] points, A is convex. □ □

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1. Let $V = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$.

(a) Prove that $B = \{f \in V : |f(t)| \leq 1 \text{ for all } t \in [0, 1]\}$ is a convex set.

Proof. Assume that $f, g \in B$, and let $h \in [f, g]$. Then $h = \alpha f + (1-\alpha)g$ for some α with $0 \leq \alpha \leq 1$. Note that since $\alpha \in [0, 1]$, $1-\alpha \in [0, 1]$ as well, and most importantly we have $\alpha \geq 0$ and $1-\alpha \geq 0$.

Assume that $t \in [0, 1]$. Then since $f, g \in B$, and by the triangle inequality, we have

$$|h(t)| = |\alpha f(t) + (1-\alpha)g(t)| \leq \alpha|f(t)| + (1-\alpha)|g(t)| \leq \alpha + (1-\alpha) = 1.$$

Therefore, $|h(t)| \leq 1$. Since $t \in [0, 1]$ was arbitrary, it follows that $h \in B$. Since $h \in [f, g]$ was arbitrary, it follows that $[f, g] \subseteq B$. Since $f, g \in B$ were arbitrary, it follows that B is a convex set. □

(b) Prove that $K = \{f \in V : f(t) \leq 0 \text{ for all } t \in [0,1]\}$ is a convex set.

Proof. Assume that $f, g \in K$, and let $h \in [f, g]$. Then $h = \alpha f + (1-\alpha)g$ for some $\alpha \in [0,1]$. Assume that $t \in [0,1]$. Since $f, g \in K$, we have $f(t) \leq 0$ and $g(t) \leq 0$, from which it follows that $\alpha f(t) \leq 0$ and $(1-\alpha)g(t) \leq 0$. Then,

$$h(t) = \alpha f(t) + (1-\alpha)g(t) \leq \alpha f(t) \leq 0.$$

Therefore, $h(t) \leq 0$. Since $t \in [0,1]$ was arbitrary, it follows that $h \in K$. Since $h \in [f,g]$ was arbitrary, it follows that $[f,g] \subseteq K$. Since $f, g \in K$ were arbitrary, it follows that K is a convex set. □

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1. Prove that the intersection of convex sets is convex.

   Proof. Consider the intersection of a collection of convex sets. This collection of convex sets can be finite, countable, or uncountable. If the intersection is empty or only contains one point, then the set is trivially convex. If the intersection contains more than one point, then consider two arbitrary points x and y in said intersection. Then $[x, y]$ must also lie in the intersection since each of the sets is convex. Therefore, the intersection itself is a convex set. □

2. If $T : \mathbb{R}^d \to \mathbb{R}^d$ is linear and $A \subseteq \mathbb{R}^d$ is convex, then $T(A)$ is convex.

   Proof. Consider $\mu, \nu \in T(A)$. Hence μ and ν can be written as $\mu = T(x)$ and $\nu = T(y)$ for $x, y \in A$. Then $\alpha\mu + (1-\alpha)\nu = \alpha T(x) + (1-\alpha)T(y) = T(\alpha x) + T((1-\alpha)y) = T(\alpha x + (1-\alpha)y)$. So $\alpha x + (1-\alpha)y \in A$ since A is convex. Thus $\alpha\mu + (1-\alpha)\nu \in T(A)$. Therefore $T(A)$ is convex. □

3. Show that the image of a polyhedron P under a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^n$ is a polyhedron.

   Proof. (Remark: The following proof is an adapted version of that found in Barvinok's text. The key ideas were found there and the discussion below provides further detail of the solution.)

   (a) **Case 1:** First, we will consider the case where the linear transformation T is invertible and maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let H_0 be a linear hyperspace through the origin. Then there exists a collection of vectors $\{v_i\}_{i=1}^{d-1}$ such that $H_0 = \text{span}\{v_1, ..., v_{d-1}\}$. Note then that for each $x \in H_0$, there exists a collection of scalars $\{a_i\}_{i=1}^{d-1}$ such that $x = \sum_{i=1}^{d-1} a_i v_i$. Since T is linear, then

   $$T(x) = \sum_{i=1}^{d-1} a_i T v_i$$

and note that since T is invertible, then Tv_i is non-zero for each $i \in \{1, ..., d-1\}$. Thus, we have that $T(H_0) = \text{span}\{Tv_1, ..., Tv_{d-1}\}$. That is to say, then $T(H_0)$ is also a hyperspace in $\mathbb{R}^n$. Furthermore, note that any hyperspace H_a can be viewed as a hyperspace through the origin shifted by some vector $a \in \mathbb{R}^n$. That is to say, $H_a = \{a + v : v \in H_0\}$. Thus, for any w in H_a, we have that $w = a + v_w$ and so $Tw = Ta + Tv_w$. Hence, observe that

$$T(H) = Ta + T(H_0).$$

But, this is just the hyperspace $T(H_0)$ shifted by some new vector $Ta \in \mathbb{R}^n$. Hence, $T(H_a)$ is also a hyperspace.

Now, since any polyhedron P can be written as the intersection of m half-spaces P_i given by $P_i = \{x \in \mathbb{R}^n : c_i \cdot x \leq \beta_i \text{ and } i \in \{1, ..., m\}\}$, we can conclude that $T(P_i)$ must also be a half-space. The general idea of this is as follows. The hyperspace which serves as the boundary of the half-space will map to a new hyperspace as shown above. And, the inequality which dictates which side of the hyperspace is included in the half-space will be preserved under the linear transformation. This can be seen via the following argument. Let $P = \cap_{i=1}^m P_i$ be defined as above and let $a_i = ((T^*)^{-1})c_i$ where T^* is the conjugate linear transformation (in this case, the transpose since our matrix is real). Notice that for every $y \in T(P)$, there exists some $x_y \in \mathbb{R}^n$ such that $Tx_y = y$, or rather, $x_y = T^{-1}y$. Consider the following:

$$\begin{aligned} a_i \cdot y &= ((T^*)^{-1})c_i \cdot y \\ &= ((T^{-1})^*)c_i \cdot y \\ &= c_i \cdot (T^{-1})y \\ &= c_i \cdot x_y \\ &\leq \beta_i. \end{aligned}$$

Therefore, we have that $T(P) = \{y \in \mathbb{R}^n : c_i \cdot y \leq \beta_i \text{ and } i \in \{1, ..., m\}\}$ which is a polyhedron in $\mathbb{R}^n$. This finishes our first case.

(Remark: Though the only necessary detail required for the proof of this case lies in the last paragraph, the prior information describes with more clarity the general idea behind the argument, yielding a more intuitive discussion of the problem at hand.)

(b) **Case 2:** Let T be any linear transformation (not necessarily invertible) from $\mathbb{R}^d$ to $\mathbb{R}^n$ for any d and n in $\mathbb{N}$. Note that if $\ker(T) = \{0\}$, then the restriction of $T : \mathbb{R}^d \to T(\mathbb{R})$ is invertible and so, by Case 1, we are done. For a general T, define a new transformation $\hat{T} : \mathbb{R}^d \to \mathbb{R}^n \oplus \mathbb{R}^d \cong \mathbb{R}^{n+d}$ given by $\hat{T}(x) = (Tx, x)$. Notice that $\ker(\hat{T}) = \{0\}$. So, from Case 1, we have that $\hat{T}(P)$ is a polyhedron in $\mathbb{R}^{n+d}$. But, we want $T(P)$ as a polyhedron in $\mathbb{R}^n$. Notice that $T(P)$ can be obtained by d projections of $\hat{T}(P)$. In particular, each projection map π_i where $i \in \{0, ..., d-1\}$ maps from $\mathbb{R}^{n+d-i}$ to $\mathbb{R}^{n+d-i-1}$ by simply removing the last component of the vector. By Lemma 9.1 from Barvinok's text, we know that this projection preserves polyhedrons. That is, for each polyhedron P in $\mathbb{R}^{n+d-i}$, the projection $\pi_i(P)$ is a polyhedron in $\mathbb{R}^{n+d-i-1}$. Thus, since $T(P)$ is the result of finitely many such projections of the polyhedron $\hat{T}(P)$, then $T(P)$ must also be a polyhedron.

$\square$

4. (a) If A and B are convex then so is $A+B$.
   (b) If A is convex and $\alpha, \beta \in \mathbb{R} : \alpha\beta > 0$ then $(\alpha+\beta)A = \alpha A + \beta A$.
   (c) Show the previous statement is false if either of its assumptions are false.

Proof. For the first two parts let A and B be convex.

(a) Let $x, y \in A+B$ and $z \in [x, y]$. Then for some $a_x, a_y \in A$, $b_x, b_y \in B$ and $\lambda \in [0,1]$, we have: $a_x + b_x = x$, $a_y + b_y = y$ and $z = \lambda x + (1-\lambda)y$. From here we can see

$$z = \lambda x + (1-\lambda)y = z = \lambda(a_x + b_x) + (1-\lambda)(a_y + b_y)$$
$$= (\lambda a_x + (1-\lambda)a_y) + (\lambda b_x + (1-\lambda)b_y) \in [a_x, a_y] + [b_x, b_y] \subseteq A+B.$$

Thus, $A+B$ is convex.

(b) Let $\alpha, \beta \in \mathbb{R} : \alpha\beta > 0$. Picking $x \in \alpha A + \beta A$, we have that $x = \alpha a + \beta b$ for some $a \in A$ and $b \in B$. Note that $sgn(\alpha) = sgn(\beta) = sgn(\alpha+\beta) \neq 0$. This necessarily gives $\frac{\alpha}{\alpha+\beta} = |\frac{\alpha}{\alpha+\beta}|$ and, similarly, $\frac{\beta}{\alpha+\beta} = |\frac{\beta}{\alpha+\beta}|$. Finally, noting that $\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1$, we have $(\alpha+\beta)(\frac{\alpha}{\alpha+\beta}a + \frac{\beta}{\alpha+\beta}a) \in (\alpha+\beta)[a, b]$ and $(\alpha+\beta)[a, b] \subseteq (\alpha+\beta)A$ since A is convex.

(c) As a counterexample if A is not convex, consider $B_n(x, r)$ to be an open n-dimensional sphere of radius r centered at x. Define $B = \overline{B_n(0,2) \setminus B_n(0,1)}$. Notice that B is not convex, and if $|\alpha|, |\beta| \in [1, \infty)$ then we have

$$(\alpha+\beta)B \neq \alpha B + \beta B = \overline{B_n(0, 2|\alpha| + 2|\beta|)}$$

In particular, notice that even if A were any set with cardinality greater than one, then taking $\alpha = -\beta \neq 0$ (i.e. $\alpha\beta < 0$) would give the statement $\{0\} \neq \alpha A + \beta A$ since $\exists a, b \in A : a \neq b$.

$\square$

6. Let $S \subset \mathbb{R}^d$ be compact and convex. Prove that there exists $\mathbf{u} \in \mathbb{R}^d$ such that $-\frac{1}{d}S \subset S + \mathbf{u}$

Proof. For each $x \in S$, let
$$A_x = \{\mathbf{u} : -\frac{1}{d}x + \mathbf{u} \in S\}$$
We show A_x is convex and compact for each $x \in S$.
Let $u_1, u_2 \in A_x$ and $0 \leq \lambda \leq 1$. We have $-\frac{1}{d}x + u_1 \in S$ and $-\frac{1}{d}x + u_2 \in S$ by definition of A_x.
Then $-\frac{1}{d}x + (1-\lambda)u_1 + \lambda u_2 = (1-\lambda)(-\frac{1}{d}x + u_1) + \lambda(-\frac{1}{d}x + u_2) \in S$ since S is convex. So A_x is convex for each x.

Further since each A_x is translation of the set S by the vector $\frac{1}{d}x$, then since S is compact, then A_x is also compact.

Now let $x_1, ..., x_{d+1} \in S$ be any set of $d+1$ distinct points of S We show $\bigcap_{i=1}^{d+1} A_{x_i} \neq \emptyset$
Let $u = \sum_{i=1}^{d+1} \frac{1}{d} x_i$. For each $j \in \{1...d+1\}$ we have $-\frac{1}{d} x_j + u = \sum_{i \neq j} \frac{1}{d} x_i \in S$, Since this is a convex combination of the vectors $x_1, ...x_{d+1}$ which are in S.
But, then we have $u \in A_{x_j}$. Since this is true for each $j \in \{1...d+1\}$, then $u \in \bigcap_{i=1}^{d+1} A_{x_i}$, so $\bigcap_{i=1}^{d+1} A_{x_i} \neq \emptyset$.
Since any intersection of $d+1$ sets from the collection $\{A_x\}_{x \in S}$ is nonempty, then by Helly's theorem, any finite intersection of sets from the same collection $\{A_x\}_{x \in S}$ is nonempty. Since each A_x is compact, then by the finite intersection property, $\bigcap_{x \in S} \{A_x\} \neq \emptyset$
Hence, there exists a $u \in \bigcap_{x \in S} \{A_x\} \neq \emptyset$. Then $-\frac{1}{d} x + u \in S$ for all $x \in S$, so $-\frac{1}{d} S \subset S - u$ and the statement holds. □

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1. Prove that $conv(conv(S)) = conv(S)$ for any $S \subset V$.

   Proof. The set $conv(conv(S))$ is convex and contains S, while the set $conv(S)$ is the smallest convex set that contains S. Hence, $conv(conv(S)) \supseteq conv(S)$.

   Similarly, the set $conv(conv(S))$ is the smallest convex containing $conv(S)$, which proves the opposite inequality. □

2. Prove that if $A \subset B$, then $conv(A) \subset conv(B)$.

   Proof. Both $conv(B)$ and $conv(A)$ are convex sets and contain A. □

3. Prove that $(conv(A) \cup conv(B)) \subset conv(A \cup B)$.

   Proof. The set $conv(A \cup B)$ is convex and contains both A and B, therefore it contains both $conv(A)$ and $conv(B)$. □

4. Let $S \subset V$ and let $u, v \in V$ be points such that $u \notin conv(S)$ and $v \notin conv(S)$. Prove that if $u \in conv(S \cup \{v\})$ and $v \in conv(S \cup \{u\})$, then $u = v$.

   Proof. Assume that $u \neq v$. Since $u \in conv(S \cup \{v\})$, $u = \lambda z_1 + (1-\lambda)v$ for some $z_1 \in S$, $\lambda \in (0,1)$. Similarly, $v = \mu z_2 + (1-\mu)u$ for some $z_2 \in S$, $\mu \in (0,1)$. Then

   $$u = \lambda z_1 + (1-\lambda)(\mu z_2 + (1-\mu)u) = \lambda z_1 + \mu(1-\lambda)z_2 + (1-\lambda)(1-\mu)u.$$

   This means that
   $$u = \frac{\lambda}{\lambda + \mu - \lambda\mu} z_1 + \frac{\mu(1-\lambda)}{\lambda + \mu - \lambda\mu} z_2,$$
   contradicting the fact that $u \notin conv(S)$. Therefore we must have $u = v$.

   □

5. (Gauss-Lucas Theorem). Let $f(z)$ be a non-constant polynomial in one complex variable z, and let $z_1, \ldots z_m$ be the roots of f. Let us interprete a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Prove that each root of the derivative $f'(z)$ lies in the convex hull of $\{z_1, \ldots z_m\}$.

   Proof. Let us assume that f has n distinct roots $z_1, \ldots, z_n$. Then $f(z) = (z - z_1) \cdots (z - z_n)$, and
$$f'(z) = \sum_{i=1}^{n} \prod_{j \neq i} (z - z_j).$$
   Assume that $f'(w) = 0$. Then $\sum_{i=1}^{n} \prod_{j \neq i} \overline{(w - z_j)} = 0$. Multiplying the last expression by $(w - z_1) \cdots (w - z_n)$ yields
$$0 = \sum_{i=1}^{n} (w - z_i) \prod_{j \neq i} \|w - z_j\|^2.$$
   Let $\alpha_i := \prod_{j \neq i} \|w - z_j\|^2$. Solving for w in the above equation, we obtain
$$w = \frac{1}{\alpha_1 + \cdots \alpha_n} \left(\alpha_1 z_1 + \cdots \alpha_n z_n \right),$$
   which means that w is the convex hull of $z_1, \ldots, z_n$.

   □

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1. Prove that $\Delta = \left\{ (\xi_1, ..., \xi_{d+1}) \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} \xi_i = 1 \text{ and } \xi_i \geq 0,\ i \in \{1, ..., d+1\} \right\}$ is a polytope n $\mathbb{R}^{d+1}$.

   Proof. For every $x \in \Delta$, note that
$$\begin{aligned} x &= \sum_{i=1}^{d+1} (0, ..., 0, \xi_i, 0, ...0) \\ &= \sum_{i=1}^{d+1} \xi_i (0, ..., 0, 1, 0, ...0). \end{aligned}$$
   Note that since $\sum_{i=1}^{d+1} \xi_i = 1$, then x is a convex combination of the points $v_i = (0, ..., 0, 1, 0, ..., 0,)$ for each $i \in \{1, ..., d+1\}$ where 1 is in the ith component. This implies that x belongs to the convex hull, $\text{conv}(V)$, where $V = \{v_i\}_{i=1}^{d+1}$. Hence, we have that $\Delta \subseteq \text{conv}(V)$. Reversing this argument, we see that $\text{conv}(V) \subseteq \Delta$. Therefore, $\Delta = \text{conv}(V)$. □

2. Prove that $I = [0,1]^n \subset \mathbb{R}^n$ is a polytope.

   Proof. Notice that for $n = 0$ this is true. Then inducting on the dimension, assume that it is true $n < m$. Then there is some set of points $\{x_1, ..., x_h\}$ whose convex hull is $[0,1]^{m-1}$. Take the points in $[0,1]^m$ given by $G = \{(0, x_1), ..., (0, x_h), (1, x_1), ..., (1, x_h)\}$. Take $x = (a, b) \in [0,1] \times [0,1]^{m-1}$, then notice that $b = \sum_{i=1}^{h} \beta_i x_i$ and

   $$(a, b) = (1-a)\sum_{i=1}^{h} \beta_i (0, x_i) + a \sum_{i=1}^{h} \beta_i (1, x_i)$$
   $$1 = \sum_{i=1}^{h} ((1-a)\beta_i + a\beta_i)$$

   Thus, $[0,1]^m \subseteq conv(G)$, further noting $[0,1]^m$ is convex and $G \subset [0,1]^m$ gives the reverse containment. Therefore $[0,1]^m = conv(G)$ and $[0,1]^m$ is a polytope for every dimension m, by induction. $\square$

3. We define a **Cross Polytope** (also Hyperoctahedron or ℓ_1 ball) in $\mathbb{R}^d$ to be the subset of $\mathbb{R}^d$:

   $$O = \{(\xi_1, \xi_2, ...\xi_d) \in \mathbb{R}^d : \sum_{i=1}^{d} |\xi_i| \leq 1\}.$$

   We prove that O is a polytope, which justifies the terminology.

   Proof. For each $i \in \{1, 2..., d\}$, let $\mathbf{e_i}$ be the ith standard unit vector. and let $\mathbf{f_i} = -\mathbf{e_i}$, and let S be the union of all these points.

   $$S = \bigcup_{i \in \{1,2...,d\}} (\{\mathbf{e_i}\} \cup \{\mathbf{f_i}\}).$$

   We show that O is exactly the convex hull of S. Because the unit vectors and negative unit vectors have ℓ_1 norm 1, they are clearly all in O.
   We now show that O is convex. If $\xi = (\xi_1, \xi_2, ...\xi_d), \mu = (\mu_1, \mu_2, ...\mu_d) \in O$ and $0 \leq \lambda \leq 1$ then we have

   $$\begin{aligned}
   \sum_{i=1}^{d} |(\lambda\xi + (1-\lambda)\mu)_i| &= \sum_{i=1}^{d} |\lambda\xi_i + (1-\lambda)\mu_i| \\
   &\leq \sum_{i=1}^{d} (\lambda|\xi_i| + (1-\lambda)|\mu_i|) \quad \text{(by triangle inequality and } 0 \leq \lambda \leq 1) \\
   &= \lambda \sum_{i=1}^{d} |\xi_i| + (1-\lambda)\sum_{i=1}^{d} |\mu_i| \\
   &\leq \lambda + (1-\lambda) \quad \text{(since } \xi, \mu \in O) \\
   &= 1.
   \end{aligned}$$

This shows that the convex combination of any two points in O is also in O so O is convex. Since $S \subseteq O$, it follows that $Conv(S) \subseteq O$. Now we show the reverse inclusion:

Let K be any convex set containing S. Since K is convex, it contains every convex combination of points in S.

In particular, $0 = (1/2)\mathbf{e_i} + (1/2)\mathbf{f_i}$, showing $0 \in K$

Suppose $\xi = (\xi_1, \xi_2, ...\xi_d) \in O$, where $\xi \neq 0$. Then let $\sigma = \sum\limits_{i=1}^{d} |\xi_i|$. then $0 < \sigma \leq 1$ by definition of O (σ nonzero if ξ is.)
Consider the combination:

$$\mu = \sum_{i=1}^{d} \left(\frac{|\xi_i|}{2\sigma} + \frac{\xi_i}{2} \right) \mathbf{e_i} + \sum_{i=1}^{d} \left(\frac{|\xi_i|}{2\sigma} - \frac{\xi_i}{2} \right) \mathbf{f_i}$$

for any $j \in \{1, 2, ..., n\}$. Projecting to the jth component yields: $\mu_j = \left(\frac{|\xi_j|}{2\sigma} + \frac{\xi_j}{2} \right) - \left(\frac{|\xi_j|}{2\sigma} - \frac{\xi_j}{2} \right) = \xi_j$. Hence, $\mu = \xi$.

We have written μ as a combination of points in S. We now show that the combination is convex. Summing the coefficients, we find:

$$\sum_{i=1}^{d} \left(\frac{|\xi_i|}{2\sigma} + \frac{\xi_i}{2} \right) + \sum_{i=1}^{d} \left(\frac{|\xi_i|}{2\sigma} - \frac{\xi_i}{2} \right) = \sum_{i=1}^{d} \frac{|\xi_i|}{\sigma} = \frac{\sigma}{\sigma} = 1$$

We now show that all the coefficients are non-negative.
Because $0 < \sigma \leq 1$, then $\frac{1}{\sigma} \geq 1$, so for all i,

$$\frac{|\xi_i|}{2\sigma} + \frac{\xi_i}{2} \geq \frac{|\xi_i| + \xi_i}{2} \geq 0,$$

and similarly:

$$\frac{|\xi_i|}{2\sigma} - \frac{\xi_i}{2} \geq \frac{|\xi_i| - \xi_i}{2} \geq 0.$$

Hence, $\xi = \mu$ is a convex combination of points in S. Thus, $\xi \in K$ for every convex K containing S, showing that $O \subseteq conv(S)$ so $O = conv(S)$ Since $S$is finite, O is a polytope, as we wished to show. ☐

4. Show that the two-dimensional unit disk $B = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^2 + \xi_2^2 \leq 1\}$ is not a polytope.

   Proof. First, we will prove the following claim:

   For $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we have $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$.

Indeed, componentwise,

$$\begin{aligned}\|x+y\|^2 &= (x_1+y_1)^2 + \ldots + (x_d+y_d)^2 \\ &= (x_1^2 + 2x_1y_1 + y_1^2) + \ldots + (x_d^2 + 2x_dy_d + y_d^2) \\ &= (x_1^2 + \ldots + x_d^2) + (y_1^2 + \ldots + y_d^2) + 2(x_1y_1 + \ldots x_dy_d) \\ &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle.\end{aligned}$$

Now, consider the d-dimensional disk B and assume that B is a polytope. That is, $B = conv(\{x_1, x_2, \ldots, x_n\})$ where $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$. Observe that any non-trivial convex combination $\sum \alpha_i x_i$ can be written as a convex combination of two points in $\text{conv}(\{x_i\}_{i=1}^n)$ as follows (WLOG assume $\alpha_n > 0$):

$$\begin{aligned}\sum_{i=1}^n \alpha_i x_i &= \left(\sum_{i=1}^{n-1} \alpha_i x_i\right) + \alpha_n x_n \\ &= (\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}) \left(\sum_{i=1}^{n-1} \frac{\alpha_i x_i}{\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}}\right) + \alpha_n x_n \\ &= \lambda \left(\sum_{i=1}^{n-1} \frac{\alpha_i x_i}{\lambda}\right) + (1-\lambda) x_n\end{aligned}$$

Since $\displaystyle\sum_{i=1}^{n-1} \frac{\alpha_i}{\lambda} = 1$, $\displaystyle\sum_{i=1}^{n-1} \frac{\alpha_i x_i}{\lambda} \in \text{conv}(\{x_i\}_{i=1}^n)$, so the above is the desired combination.

Now, for any two distinct points $a, b \in \text{conv}(\{x_i\}_{i=1}^n)$, we have $\|a\| \leq 1$ and $\|b\| \leq 1$ (and thus the projection of one onto the other must have norm less than 1). Consider the norm of any convex combination $\lambda a + (1-\lambda)b, \lambda > 0$:

$$\begin{aligned}\|\lambda a + (1-\lambda)b\|^2 &= \|\lambda a\|^2 + \|(1-\lambda)b\|^2 + 2\langle \lambda a, (1-\lambda)b\rangle \\ &= \lambda^2\|a\|^2 + (1-\lambda)^2\|b\|^2 + 2(\lambda)(1-\lambda)\langle a, b\rangle \\ &\leq \lambda^2 + (1-\lambda)^2 + 2(\lambda)(1-\lambda)\langle a, b\rangle \\ &< \lambda^2 + (1-\lambda)^2 + 2(\lambda)(1-\lambda) \\ &= (\lambda + (1-\lambda))^2 \\ &= 1\end{aligned}$$

Thus any non-trivial convex combination of $\{x_i\}_{i=1}^n$ has norm strictly less than 1. Since there are finitely many x_i but infinitely many points in B with norm 1, there must exist a point in B that is not in $\text{conv}(\{x_i\}_{i=1}^n)$. Therefore B is not a polytope. $\square$

5. Let $V = C[0,1]$. Let $A = \{f \in V : 0 \leq f(t) \leq 1 \text{ for all } t \in [0,1]\}$. Prove that A is not a polytope.

   Proof. Assume the contrary, and suppose that A is a polytope. Then there exists $f_1, ..., f_n \in A$ such that $conv(\{f_1, ..., f_n\}) = A$. Let $f_* : [0,1] \to \mathbb{R}$ be defined by $f_*(t) = 1$ for all $t \in [0,1]$ (this is defined mainly for convenience as will be seen later, since $f_* \in A$).

   Assume $f \in C[0,1]$. Then since $[0,1]$ is compact, we see that there exists $a, b \in \mathbb{R}$ such that $a = \min_{t\in[0,1]} f(t)$ and $b = \max_{t\in[0,1]} f(t)$. Then we may view f as $f : [0,1] \to [a,b]$.

   If f is a constant function, then there exists $d \in \mathbb{R}$ such that $f(t) = d$ for all $t \in [0,1]$. Then we see that $f(t) = d = d \cdot 1 = d \cdot f_*(t)$ for all $t \in [0,1]$.

   Now we assume that f is not a constant function. Then it follows that $a \neq b$. Define $\phi : [a,b] \to [0,1]$ by $\phi(t) = \frac{t-a}{b-a}$ for $t \in [a,b]$. Then it follows from basic real analysis that ϕ is continuous and bijective. Similarly we see that $\phi^{-1} : [0,1] \to [a,b]$ defined by $\phi^{-1}(t) = (b-a)t + a$ is also continuous and bijective. With these, we see that $f(t) = (\phi^{-1} \circ \phi \circ f)(t)$ for all $t \in [0,1]$ since ϕ^{-1} is the inverse function of ϕ.

   From it's construction, we see that $0 \leq (\phi \circ f)(t) \leq 1$ for all $t \in [0,1]$, meaning $\phi \circ f \in A$. Then, since $A = conv\{f_1, ..., f_n\}$, we see that there exists $\alpha_k \in [0,1]$ for each $k \in \{1, .., n\}$ such that $\sum_{k=1}^n \alpha_k = 1$ and $(\phi \circ f)(t) = \sum_{k=1}^n \alpha_k f_k(t)$ for all $t \in [0,1]$. With this, we see

$$\begin{aligned}
f(t) &= (\phi^{-1} \circ \phi \circ f)(t) \\
&= \phi^{-1}((\phi \circ f)(t)) \\
&= \phi^{-1}(\sum_{k=1}^{n} \alpha_k f_k(t)) \\
&= (b-a)\left(\sum_{k=1}^{n} \alpha_k f_k(t)\right) + a \\
&= \left(\sum_{k=1}^{n} \alpha_k (b-a) f_k(t)\right) + a \cdot f_*(t).
\end{aligned}$$

   Therefore f is a linear combination of $\{f_*, f_1, .., f_n\}$. Since $f \in V$ was arbitrary, we may do this for every such f, meaning $C[0,1] = span\{f_*, f_1, ..., f_n\}$. Since $C[0,1]$ is spanned by a finite number of vectors (functions), it follows that $C[0,1]$ is a finite dimensional vector space, contradicting a popular fact from functional analysis that $C[0,1]$ is an infinite dimensional vector space. Therefore it must be that A is not a polytope. □

Page 12.

1. Give an example of a closed set in $\mathbb{R}^2$ whose convex hull is not closed.

2. Prove that the convex hull of an open set in $\mathbb{R}^d$ is open.

   Proof. Let $A \subseteq \mathbb{R}^d$ be open. Let $\mu \in conv(A)$. Then $\mu = \sum\limits_{i=1}^{m} \alpha_i x_i$, for $0 \leq \alpha_i \leq 1$, $\sum\limits_{i=1}^{m} \alpha_i = 1$. For each $i \in 1, \ldots, m$, there exists and $r_i > 0$ such that $B(x_i, r_i) \subset A$. Pick $\epsilon \leq \min_{0 \leq i \leq m} r_i$.

For $\nu \in B(0,\epsilon)$, then each $x_i+\nu \in A$. Thus, $\mu+\nu = \sum_{i=1}^{m} \alpha_i x_i + \sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \alpha_i(x_i+\nu) \in conv(A)$. Thus $conv(A)$ is also open. $\square$

Page 20.

1. Let $\{A_i\}_{i<m} \subset \mathbb{R}^d$ be a collection of convex sets. Prove that if every $k \leq d+1$ of the sets have a common point, then for all $d-(k-1)$-dimensional subspaces, L, there is a $u \in \mathbb{R}^d$ such that $L+u$ intersects all the A_i's.

   Proof. Consider the orthogonal projection onto $L^\perp$ given by $\Pi : \mathbb{R}^d \to L^\perp \subset \mathbb{R}^d$. Noting that $L^\perp$ is a $k-1$ dimensional subspace of $\mathbb{R}^d$, there is an isomorphism $h : L^\perp \to \mathbb{R}^{k-1}$. Now $(h \circ \Pi)(\cap_{i=1}^{k} A_{\alpha_i}) = \cap_{i=1}^{k}(h \circ \Pi)(A_{\alpha_i}) \neq \emptyset$, by assumption hence $\exists u \in \cap_{i=1}^{m}(h \circ \Pi)(A_i) = h(\cap_{i=1}^{m}\Pi(A_i))$ Helly's Theorem. Since h is an isomorphism, h^{-1} exists and $h^{-1}(u) \in \Pi(\cap_{i=1}^{m} A_i)$. This gives the desired statement as $\emptyset \neq ((\Pi^{-1} \circ h^{-1})(u)) \cap A_i = (h^{-1}+L) \cap A_i$. $\square$

2. Let $A_1, \ldots, A_m$, and C be convex sets in $\mathbb{R}^d$ such that for every $d+1$ of the A_i sets there is a translate $C+u$ with $u \in \mathbb{R}^d$ that intersects them. Then there exists $v \in \mathbb{R}^d$ such that $C+v$ intersects A_i for all i.

   Proof. Define $W_i = \{x : (x+C) \cap A_i \neq \emptyset\}$. If some $d+1$ of the A_i are intersected by $C+u$, then u is an element in the intersection of the corresponding W_i. That is, any intersection of $d+1$ of the W_i is nonempty. Now, to show that the W_i are convex, let $x, y \in W_i$ for some i. Then, by definition of W_i, there exists $c_x, c_y \in C$ such that $x+c_x \in A_i$ and $y+c_y \in A_i$. Since A_i is convex:

   $$\begin{aligned} [x+c_x, y+c_y] &= \lambda(x+c_x) + (1-\lambda)(y+c_y) && \in A_i \\ &\implies (\lambda x + (1-\lambda)y) + (\lambda c_x + (1-\lambda)c_y) && \in A_i. \end{aligned}$$

   But since C is convex, $(\lambda c_x + (1-\lambda)c_y) = [c_x, c_y] \in C$ and thus, by definition, $(\lambda x + (1-\lambda)y) = [x, y] \in W_i$. Therefore W_i is convex for all i.

   Now, applying Helly's Theorem, we get that $\bigcap_{i=1}^{m} W_i = \{x : (x+C) \cap A_i \neq \emptyset \, \forall i\}$ is nonempty. Suppose v is an element of the intersection. Then the translate $C+v$ intersects every A_i.

   $\square$

3. Let $\mathcal{A} = \{A_1, ..., A_m\}$ and C be convex sets in $\mathbb{R}^d$ such that for every collection $\{A_i\}_{i=1}^{d+1} \subseteq \mathcal{A}$ of $d+1$ sets in $\mathcal{A}$, there is a translate $C+u$ with $u \in \mathbb{R}^d$ that contains them. Show that then there exists $v \in \mathbb{R}^d$ such that $C+v$ contains all sets in $\mathcal{A}$.

   Proof. Note that if $m \leq d+1$ we are done. Suppose that $m > d+1$. For all $i \in \{1, ..., m\}$, define

   $$U_i = \{u \in \mathbb{R}^d : A_i - u \subseteq C\}.$$

Note that for any $d+1$ of these sets U_i, there must exist some $u \in \bigcap_{k=1}^{d+1} U_{i_k}$, namely the u such that $C+u$ covers $\bigcup_{k=1}^{d+1} A_{i_k}$. Now, we wish to apply Helly's Theorem to these sets U_i. However, to do this, we must first show that each U_i is convex.

Pick $v, w \in U_i$. Note then that $A_i - v \subseteq C$ and $A_i - w \subseteq C$. Thus we have that

$$A_i \subseteq (C+v) \cap (C+w).$$

Hence, $(C+v) \cap (C+w)$ is nonempty, so let $z \in (C+v) \cap (C+w)$. Then $z = c_0 + v = c_1 + w$ for some $c_0, c_1 \in C$. So, we have that

$$\begin{aligned} z &= \lambda(c_0 + v) + (1-\lambda)(c_1 + w) \\ &= [\lambda c_0 + (1-\lambda)c_1] + [\lambda v + (1-\lambda)w] \end{aligned}$$

and therefore z belongs to $C + (\lambda v + (1-\lambda)w)$. Since this is true for all z in $(C+v) \cap (C+w)$, then we have that $A_i \subseteq C + (\lambda v + (1-\lambda)w)$ or more importantly, $A_i - (\lambda v + (1-\lambda)w) \subseteq C$. This implies that $(\lambda v + (1-\lambda)w)$ belongs to U_i. Therefore U_i is convex for every $i \in \{1, ..., m\}$.

Now, we apply Helly's Theorem to these sets U_i to see that there exists some $u_0 \in \bigcap_{i=1}^{m} U_i$.

Finally, we then have that $C + u_0$ contains $\bigcup_{i=1}^{m} A_i$. $\square$

Page 22.

1. Assume that C is a convex set $C \subseteq \bigcup_{i=1}^{m} H_i^+$, H_i^+ half-spaces in $\mathbb{R}^d$. Show that there are $d+1$ of these half-spaces that cover C.

   Proof. Suppose that any collection of $d+1$ half-spaces does not cover C, which we can reorder if desired. Then $C \nsubseteq \bigcup_{i=1}^{d+1} H_i^+$. Consider $\{C \cap \mathbb{R}^d - H_i^+\}_{i=1}^m$.
   Then $C \cap \mathbb{R}^d - \bigcup_{i=1}^{d+1} H_i^+ \neq \emptyset \Rightarrow C \bigcap_{i=1}^{d+1} \left(\mathbb{R}^d \setminus H_i^+\right) \neq \emptyset \Rightarrow \bigcap_{i=1}^{d+1} C \cap (\mathbb{R}^d \setminus H_i^+) \neq \emptyset$. So by Helly's Theorem, there is at least a point in common, which is a contradiction. $\square$

2. Let $I_1, \ldots, I_m$ be parallel line segments in $\mathbb{R}^2$ such that for every three of them, there is a straight line that intersects all three. Prove that there is a straight line that intersects all the segments.

   Proof. Assume the segments are vertical, let one of them be $I = \{(x_0, y) : y_0 \leq y \leq y_1\}$. Then the line $y = ax + b$ intersects I if and only if $y_0 \leq ax_0 + b \leq y_1$, or equivalently, if

   $$-ax_0 + y_0 \leq b \leq -ax_0 + y_1.$$

On the (a,b)-plane, $b = -ax_0 + y_i$ are lines with slope $-x_0$ and y-intercept y_i. The set of all the lines $y = ax + b$ intersecting I corresponds to the strip

$$S_I = \{(a,b) \in \mathbb{R}^2 : -ax_0 + y_0 \leq b \leq -ax_0 + y_1\},$$

which is convex. If a line intersects three intervals $I_{i_1}, I_{i_2}, I_{i_3}$, then there exists $(\widetilde{a}, \widetilde{b}) \in S_{I_{i_1}} \cap S_{I_{i_2}} \cap S_{I_{i_3}}$. Then, by Helly's theorem, there is a point $(p,q) \in \bigcap_{i=1}^{m} S_{I_i}$, but this is equivalent to the fact that the line $y = px + q$ intersects all the intervals. $\square$

3. Let $A_1, ..., A_m$ be convex sets in $\mathbb{R}^2$ such that for every pair of sets there is a line parallel to the x-axis that intersects them. Show that there is a line parallel to the x-axis intersecting all of them.

Proof. By hypothesis we see that for each $i,j \in \{1,...,m\}$ with $i \neq j$, there exists $y_{i,j} \in \mathbb{R}$ such that the line $L_{i,j} = \{(x, y_{i,j}) : x \in \mathbb{R}\}$ intersects A_i and A_j. This means there exists $x_i, x_j \in \mathbb{R}$ such that $(x_i, y_{i,j}) \in A_i \cap L_{i,j}$ and $(x_j, y_{i,j}) \in A_j \cap L_{i,j}$.

For each $j \in \{1,..,m\}$ let $A_j^y = \{b \in \mathbb{R} | \exists a \in \mathbb{R} : (a,b) \in A_j\}$ (i.e. $A_j^y = \pi_2(A_j)$, where $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is defined by $\pi_2((x,y)) = y$). Then by the above we see that for $i,j \in \{1,...,m\}$ with $i \neq j$, we have $y_{i,j} \in A_i^y \cap A_j^y$.

We will now show that A_j^y is convex for $j \in \{1,...,m\}$. To that end, fix j and assume $p,q \in A_j^y$. Then there exists $u,v \in \mathbb{R}$ such that $(u,p),(v,q) \in A_j$. Since A_j is convex, for each $t \in [0,1]$ the point $t(u,p) + (1-t)(v,q) \in A_j$. For any $t \in [0,1]$ we then see that $t(u,p) + (1-t)(v,q) = (tu, tp) + ((1-t)v, (1-t)q) = (tu + (1-t)v, tp + (1-t)q)$. This implies that $tp + (1-t)q \in A_j^y$. Since $p,q \in A_j^y$ was arbitrary, it follows that this is true for all such p and q, meaning A_j^y is a convex subset of $\mathbb{R}^1$.

If $m = 1$, then the claim is immediately true with the line intersecting all of them being any line through A_1.

If $m \geq 2$, then we see that for every two subsets A_i^y and A_j^y of $\mathbb{R}^1$, A_i^y and A_j^y contain the point $y_{i,j}$. Then by Helly's Theorem we see that $\bigcap_{i=1}^m A_i^y \neq \emptyset$. With that, we see that there exists $y \in \mathbb{R}$ such that $y \in A_i^y$ for each $i \in \{1,...,m\}$.

Let $L = \{(x,y) : x \in \mathbb{R}\}$. Then since $y \in A_j^y$ for each $j \in \{1,...,m\}$ there exists $z_j \in \mathbb{R}$ such that $(z_j, y) \in A_i$. Then it follows that from the construction of L that $(z_j, y) \in L$ as well. Therefore L intersects A_j for each $j \in \{1,...,m\}$. From its construction we see that L is parallel to the x-axis (i.e. it is horizontal), and so L is the desired line. $\square$

Chapter 2: Faces and Extreme Points

Page 50.

1. If F is a face of K, then F is a closed convex set. Also, if K is compact so is F.

Proof. If F is such a face, then by definition, there exists an isolating hyperplane H such that $F = K \cap H$. We have seen before that hyperplanes are convex sets, and that the intersection of

convex sets is convex, therefore, F is convex. Also, it is clear that hyperplanes are closed sets, since if $H = \{x \in \mathbb{R}^n : \langle a, x\rangle = b\}$, let $f : \mathbb{R}^n \to \mathbb{R}$ be the continuous function $f(x) = \langle a, x\rangle$. Then $H = f^{-1}(\{b\})$ where $\{b\} \in \mathbb{R}$ is closed. Since intersection of closed sets are closed, F is also closed.
If K is compact. Then $F = K \cap H$ is a closed subspace of a compact space, and therefore is compact, by the familiar topological fact. □

5. If F_1, F_2 are faces, with $F_1 \cap F_2 \neq \emptyset$, then $F_1 \cap F_2$ is a face.

   Proof. Since F_1 and F_2 are faces, there exist hyperplanes H_1, H_2 such that $F_1 = K \cap H_1$, $F_2 = K \cap H_2$ where H_1 and H_2 are hyperplanes that isolate K. Since $F_1 \cap F_2 \neq \emptyset$, we can assume without loss of generality (Since otherwise we can just translate the hyperplanes), that $0 \in F_1 \cap F_2 \subset H_1 \cap H_2$ and hence we can write, $H_1 = \{x : \langle a, x\rangle = 0\}, H_2 = \{x : \langle b, x\rangle = 0\}$ where $a, b \in \mathbb{R}^n$

   Since H_1 and H_2 are hyperplanes that isolate K, We have various cases: We will consider only one, the others are similar. First suppose $K \subset \overline{H_1^-}$ $K \subset \overline{H_2^-}$ In this scenario, define $H = \{x : \langle a + b, x\rangle = 0\}$ We will show that $K \cap H = F_1 \cap F_2$

   To see this, first suppose $x \in F_1 \cap F_2$, then in particular, $x \in H_1 \cap H_2$, so $\langle a, x\rangle = 0$ and $\langle b, x\rangle = 0$, whereby linearity in the first component, yields $\langle a + b, x\rangle = 0$ So $X \in H \cap K$. Conversely, if, $X \in H \cap K$, then since $x \in K$, we have $x \in \overline{H_1^-}$ and $x \in \overline{H_2^-}$. Equivalently, $\langle a, x\rangle \leq 0$ and $\langle b, x\rangle \leq 0$, but $\langle a, x\rangle + \langle b, x\rangle = \langle a + b, x\rangle = 0$, so this forces $\langle a, x\rangle = 0$ and $\langle b, x\rangle = 0$. So $x \in H_1 \cap H_2$, but also $x \in K$, so $x \in F_1 \cap F_2$

   Further for any $x \in K$, we have $\langle a, x\rangle \leq 0$ and $\langle b, x\rangle \leq 0$, so $\langle a + b, x\rangle \leq 0$, showing that $K \subset \overline{H^-}$, so that H isolates K.

   The other cases are similar; However, for 2 cases, one must define $H = \{x : \langle a - b, x\rangle = 0\}$ instead. □

Page 52.

1. Let $K \subseteq \mathbb{R}^d$ be closed and convex and let $F \subseteq K$ be a face of K. Show that if u is an extreme point of F, then u is an extreme point of K.

   Proof. Since F is a face of K, then there exists some hyperspace H of $\mathbb{R}^d$ such that $H \cap K = F$, and such that for every $x \in K$, then $\langle x, c\rangle \leq \beta$ for some fixed $\beta \in \mathbb{R}$ and $c \in \mathbb{R}^d$, where equality is achieved when $x \in H$.

   Assume that we have some a and b in K such that $\frac{a+b}{2} = u$. Note that since u is in F, then u belongs to H and thus, $\langle u, c\rangle = \beta$. By properties of the inner product, then we have the following

$$\begin{aligned}\langle \frac{a+b}{2}, c\rangle &= \beta\\ \frac{1}{2}\langle a, c\rangle + \frac{1}{2}\langle b, c\rangle &= \beta\\ \langle a, c\rangle + \langle b, c\rangle &= 2\beta\end{aligned}$$

And so we see that since b belongs to K, then $\langle b, c\rangle \leq \beta$ and so $\langle a, c\rangle = 2\beta - \langle b, c\rangle \geq \beta$. Therefore $\langle a, c\rangle \leq \beta$. Similarly, we see that $\langle b, c\rangle \leq \beta$. Now, since a and b are in K, we have that $\beta \leq \langle a, c\rangle \leq \beta$ and $\beta \leq \langle b, c\rangle \leq \beta$ and so $\langle a, c\rangle = \beta = \langle b, c\rangle$. That is, a and b must belong to H. Hence, a and b belong to $H \cap K = F$ and since u is an extreme point of F, then a and b must both equal u. Therefore, since a and b were arbitrarily chosen from K such that $\frac{a+b}{2} = u$, then u must be an extreme point of K.

□

2. Let K be a compact convex subset of $\mathbb{R}^d$. Let $u \in K$ be such that $\|u\|_2 \geq \|v\|_2$ for every $v \in K$. Then u is an extreme point of K.

   Proof. Assume $a, b \in K$ satisfy $u = \frac{a+b}{2}$.

   First we will show that $\|a\| = \|b\| = \|u\|$. To that end, suppose the contrary and assume that one of $\|a\|$ or $\|b\|$ is not equal to $\|u\|$. Without loss of generality assume that $\|a\| \neq \|u\|$. Then since $a \in K$, we see that $\|a\| < \|u\|$, so that

   $$\|u\| = \left\|\frac{a+b}{2}\right\| \leq \frac{\|a\|}{2} + \frac{\|b\|}{2} \leq \frac{\|a\|}{2} + \frac{\|u\|}{2} < \frac{\|u\|}{2} + \frac{\|u\|}{2} = \|u\|,$$

   which means $\|u\| < \|u\|$, which is impossible. Therefore $\|a\| = \|u\|$, and similarly we find that $\|b\| = \|u\|$.

   Therefore we see that a, b, and u all lie on the circle of radius $\|u\|$ centered at 0.

   If a, u, and b were distinct points on this circle, then it follows by Joe's result in the second problem set that u would actually be in the interior of the circle. This means that u is both on the circle and in the interior of the circle, which is impossible.

   Therefore it must be that there exists some equalities amongst a, b, and c, meaning at least one of the equalities $a = b$, $u = a$, or $u = b$ must hold.

   If $a = b$, then we see

   $$u = \frac{a+b}{2} = \frac{a+a}{2} = a,$$

   so that $u = a = b$.

   Now assume that u is equal to at least one of a or b. Without loss of generality, assume that $u = a$. Then since we also have $u = \frac{a+b}{2}$, it follows that

   $$b = 2u - a = 2u - u = u,$$

   so that $b = u$ as well. Therefore we see that $a = b = u$. Since we have exhausted every case, and since $a, b \in K$ were arbitrary, it follows that u is an extreme point of K. □

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1. Let K be a closed convex set in $\mathbb{R}^d$. If K contains no straight line, then K contains an extreme point. (This is Lemma 3.5 in the book).

   Proof. Induct on d.

   Base case: $d = 1$. A set K satisfying the conditions is either a closed ray $[a, \infty), (-\infty, a]$ or a closed interval $[a, b]$. In each of these cases, a is an extreme point of K.

   Suppose the statement holds for closed convex sets of dimension $d - 1$ or less. Let K be a closed convex set in $\mathbb{R}^d$ containing no straight line. If K has no interior, we are done by the hypothesis. Suppose K has interior and let $u \in \partial K$. If u is an extreme point, we are done. Otherwise there exists a hyperplane H separating (not strictly) u and K. Then $F = H \cap K$ is a face of K and is therefore closed and convex. Additionally, since K contains no straight lines, F contains no straight lines. Since F has dimension less than or equal to $d - 1$, F contains an extreme point by the hypothesis. Since extreme points of F are extreme points of K, we have the conclusion.

   □

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Let $P \subseteq \mathbb{R}^d$ be a polyhedron.

1. Prove that P has finitely many faces.

2. Prove that a face of P is a polyhedron.

Proof. First let $\{v_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and $\{c_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ then define the arbitrary polyhedron defined by n vectors as

$$P_n := \{x \in \mathbb{R}^d : \quad < x, v_i > \leq c_i \text{ for all } i \leq n\}$$

Then the first claim is easy as any face is given by $P_n \cap (v^\perp + cv)$ for some $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Now note that

$$P_n \cap (v^\perp + cv) = \{x \in \mathbb{R}^d : \quad < x, v_i > \leq c_i \text{ for all } i \leq n, \ < x, v > \leq c, \text{ and } \ < x, -v > \leq -c\}$$

Since this is a polyhedron by definition, claim (1) follows. As for claim (2), let $f = P_{n+1} \cap (v^\perp - cv)$ be an arbitrary face of P_{n+1}, then either $f \cap (v_{n+1}^\perp - c_{n+1}v_{n+1}))$ is empty or it is not. If this intersection is empty then it follows that f is a face of P_n, and if it is not empty then we must have $f \cap (v_{n+1}^\perp - c_{n+1}v_{n+1}))$ as a face of $P_{n+1} \cap (v_{n+1}^\perp - c_{n+1}v_{n+1}))$ by definition, which by (1) is a polyhedron. Now letting $\#_f$ denote the number of faces of a polyhedron. We have

$$\#_f P_{n+1} \leq \#_f P_n + \#_f(P_{n+1} \cap (v_{n+1}^\perp - c_{n+1}v_{n+1})))$$

Looking closely at the formula we get a bound on the number of faces of a polyhedron based on the number of vectors defining the polyhedron and the dimension of the polyhedron (note the right-most term is really a $d - 1$ dimensional polyhedron). Since $\#_f P_1 = 1$ regardless of dimension and if $d = 1$, then $\#_f P_n \leq 3$, by finite induction using the above inequality we see $\#_f P_n$ is finite, as desired. □

Chapter 4: Polarity, Duality and Linear Programming

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Recall that for $A \subset \mathbb{R}^d, A \neq \emptyset$. The polar of A is $A^\circ = \{c \in \mathbb{R}^d | \langle c, x\rangle \leq 1, \forall x \in A\}$.

1. Prove that A° is closed, convex, and $0 \in A^\circ$.

   Proof. The difinition of the polar of A gives that the inner product is less than or equal to 1. Hence, all the limit points are included. So A° is closed. For $c_1, c_2 \in A^\circ$, $\langle c_1, x\rangle \leq 1$ and $\langle c_2, x\rangle \leq 1$. Then, for $\alpha \in [0,1]$, $\langle \alpha c_1 + (1-\alpha)c_2, x\rangle = \alpha\langle c_1, x\rangle + (1-\alpha)\langle c_2, x\rangle \leq 1$. Hence, A° is convex. Notice that $\langle 0, x\rangle = 0$, $\forall x \in A$. Thus, $0 \in A^\circ$. □

6. If $L \subset \mathbb{R}^d$ is a linear subspace, then $L^\circ = L^\perp$.

   Proof. Essentially, we want to show that $\langle c, x\rangle \leq 1 \iff \langle c, x\rangle = 0$.
   ($\Leftarrow$) Done.
   ($\Rightarrow$) Assume $\langle c, x\rangle = \alpha$ for $x \in L$. Since L is a linear subspace, then for any $\lambda \in \mathbb{R}$, $\lambda x \in L$. Hence, $\langle c, \lambda x\rangle = \lambda\langle c, x\rangle = \lambda\alpha$. But for $\lambda\alpha \leq 1$ for all $x \in L$, then $\alpha = 0$. □

8. Show $A \subseteq (A^\circ)^\circ$.

   Proof. Notice that $(A^\circ)^\circ = \{c \in \mathbb{R}^d | \langle c, x\rangle \leq 1, \ \forall x \in A^\circ\}$. Let x be arbitrary in A. Then, $\forall c \in A^\circ$, $\langle c, x\rangle \leq 1$. By definition of $A^\circ, x \in (A^\circ)^\circ$. This holds for every $x \in A$, so $A \subseteq (A^\circ)^\circ$. □

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4. Prove that the cube $I_d = [-1, 1]^d$ is self dual in dimensions 1 and 2 but not in dimensions $d \geq 3$.

   (For the higher dimensional cases, we will consider the following lemma.)

   Lemma 1. *Each vertex of a regular polyhedron P centered at zero corresponds to a face of its polar P°.*

   Proof. Let $P = \mathrm{conv}\{v_1, ..., v_n\}$ be a regular polyhedron in $\mathbb{R}^d$ with vertices $v_1, ..., v_n$. Notice then that $\{v_i\}^\circ = \{x \in \mathbb{R}^d : \langle v_i, x\rangle \leq 1\}$ defines a closed half-space H_i for all $i \in \{1, ..., n\}$. Note that for every p in P, then $p = \sum_{i=1}^n \alpha_i v_i$ such that $\sum_{i=1}^n \alpha_i = 1$. And so, observe that

$$\begin{aligned} P^\circ &= \{x \in \mathbb{R}^d : \langle p, x\rangle \leq 1 \text{ for all } p \in P\} \\ &= \left\{x \in \mathbb{R}^d : \sum_{i=1}^n \alpha_i\langle v_i, x\rangle \leq 1 \text{ for all } p = \sum_{i=1}^n \alpha_i v_i \in P\right\} \\ &\supseteq \left\{x \in \mathbb{R}^d : \langle v_i, x\rangle \leq 1 \text{ for all } i \in \{1, ..., n\}\right\} \\ &= \bigcap_{i=1}^n H_i. \end{aligned}$$

Therefore we have that $P^\circ \supseteq \bigcap_{i=1}^{n} H_i$. Furthermore, notice that for any x in P°, we have that $\langle p, x\rangle \leq 1$ for every $p \in P$. Specifcally, this is true for any v_i, $i \in \{1, ..., n\}$. And so we see that for each $i \in \{1, ..., n\}$, then $\langle v_i, x\rangle \leq 1$ and so x belongs to H_i. Thus, x belongs to $\bigcap_{i=1}^{n} H_i$ and so $P^\circ \subseteq \bigcap_{i=1}^{n} H_i$. Therefore, $P^\circ = \bigcap_{i=1}^{n} H_i$ and each vertex v_i corresponds to a unique half-space H_i which defines a side of P°. □

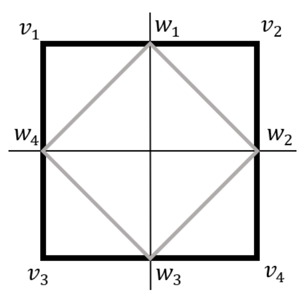
Now, let us proceed with the solution to the first part of the original problem. First, we will consider I_1 which only requires a quick observation. For the polygon I_2, we will show a claim using the lemma, and proceed to define an explicit invertible linear transformation. We shall then consider I_d where $d \geq 3$. This will be shown by contradiction, using the above lemma.

Proof. First consider the polar of $I_1 = [-1, 1]$. We see that

$$I_1^\circ = \{c \in \mathbb{R} : cx \leq 1 \text{ for all } x \in I_1\}.$$

Furthermore, since for any $x \in I_1$ then we have $-1 \leq x \leq 1$. It suffices to consider only these endpoints. So, we note that when $x = 1$, then for c to belong to I_1°, we must have $c \cdot (1) \leq 1$ and $c \cdot (-1) \leq 1$. That is to say, $-1 \leq c \leq 1$. Therefore, we have that $I_1^\circ = I_1$ and so, the identity map is an invertible linear transformation between I_1 and I_1°. Hence, I_1 is self dual.

Next, consider the polar of $I_2 = [-1, 1] \times [-1, 1]$. Observe that this cube has four vertices (and four faces). Call these vertices v_i, labeled clockwise for $i \in \{1, 2, 3, 4\}$. From the lemma, we see that P° must then have four faces corresponding to each vertex. But, since it has four sides and is a polygon in $\mathbb{R}^2$, it must also have four vertices, call them w_i, labeled clockwise for $i \in \{1, 2, 3, 4\}$. The labeling is shown in the following figure.



We notice in particular that the polar of P should be the convex hull $\text{conv}\{w_1, w_2, w_3, w_4\}$. So, we need to find a linear transformation which takes v_i to w_i for each $i \in \{1, 2, 3, 4\}$. Consider the matrix

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

This matrix T rotates the original I_2 and dilates the result by a factor of $\frac{1}{\sqrt{2}}$. Notice further that $\det T = \frac{1}{2}$ which is nonzero and hence T is an invertible linear transformation. Therefore, we have that since $T(I_2)$ is indeed I_2°, then I_2 is self dual.

Consider now the case where $d \geq 3$. Recall that any hypercube of dimension d has $2d$ faces and 2^d vertices. Recall also that since I_d is convex, closed, and containing 0, then $(I_d^\circ)^\circ = I_d$ by the Bipolar Theorem. This implies that the number of vertices of I_d° must be the equal to the number of faces of I_d. Hence, I_d° must have $2d$ vertices. Notice also that if T is an invertible linear transformation, then T must take vertices to vertices. That is, since I_d has 2^d vertices, then $T(I_d)$ must also have 2^d vertices. So, if T is an invertible linear transformation such that $T(I_d) = I_d^\circ$, then we must have that I_d° has 2^d vertices. This is a contradiction, since for $d \geq 3$, then $2d \neq 2^d$. Therefore, I_d is not self dual when $d \geq 3$. □

5. Prove that a regular polygon centered at the origin in $\mathbb{R}^2$ is self dual.

Proof. Let P be regular n-polygon in $\mathbb{R}^2$ centered at the origin. We will show that P must be self dual. First, notice that in $\mathbb{R}^2$, any regular polygon must have the same number of faces as vertices. Label the vertices of P as v_i in clockwise order for $i \in \{1, ..., n\}$. Let us also denote $w_i = \frac{v_i + v_{i+1}}{2}$ for all $i \in \{1, ..., n-1\}$ and $w_n = \frac{v_n + v_1}{2}$ to be the face vectors of P. Assume for now that P is such that the norm of each face vector w_i is equal to 1. Then we see that these face vectors will be the vertices of the polar set P°. Notice that our invertible linear transformation must first rotate P by $\frac{\pi}{n}$ radians and then dilate the resulting polygon by a factor of $\frac{1}{\|v_i\|}$. Therefore, in the case where $\|w_i\| = 1$, then we want the transformation $T = \frac{1}{\|v_i\|}R$ where R is the rotation matrix

$$R = \begin{bmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{bmatrix}.$$

This matrix is clearly invertible since it has determinant 1 which is not equal to 0. Furthermore, multiplying by the scalar $\frac{1}{\|v_i\|}$ still yields an invertible linear transformation. Now, in the general case where $\|w_i\|$ is not necessarily 1, this requires simply scaling again by $\frac{1}{\|w_i\|} = \frac{1}{\|v_i\|\cos\frac{\pi}{n}}$. That is to say, then the invertible linear transformation would be $T = \frac{1}{\|v_i\|^2\cos\frac{\pi}{n}}R$. Notice that P° must at least contain the convex hull of the images of the vertices $T(v_i)$. The above lemma assures that since the number of faces of P is equal to the number of vertices of P, then these must in fact be all the vertices of P°. Hence, we have that $T(P) = P^\circ$. Therefore, P must be self dual. □

6. Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal basis in $\mathbb{R}^4$.

Let $P = conv(e_i + e_j, e_i - e_j, -e_i - e_j)$ For all $1 \leq i \neq j \leq 4$ Prove that P is self-dual.

Proof. Let

$$Q = conv(\pm e_1, \pm e_2, \pm e_3, \pm e_4, \frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2}).$$

We will show both that Q is the polar of P and that Q can be obtained from P by a linear transformation, and then the result follows.

To see that Q is contained in the polar of P, it is enough to show that the inner product of any element in the generating set for P with an element in the generating set for Q is less or equal to 1. Then it will be true for all points in P and all points in Q since they are

just convex combinations of the ones in the generating set.
If $i \neq j$, then

$$\langle \pm e_i \pm e_j, \pm e_k \rangle = 0 \leq 1 \text{ if } k \notin \{i, j\}$$

If $k = i$ or $k = j$, then

$$\langle \pm e_i \pm e_j, \pm e_k \rangle \in \{1, -1\} \text{ is bounded above by 1.}$$

Finally

$$\left\langle \pm e_i \pm e_j, \frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2} \right\rangle = \left\langle \pm e_i \pm e_j, \frac{\pm e_i \pm e_j}{2} \right\rangle \in \{-1, 0, 1\},$$

which is again bounded by 1. Since we did this with every possible combination of generators, then $Q \subseteq P^{\text{o}}$

Now we look at the converse; Suppose $(x, y, z, w) \in P^{\text{o}}$. Since Q is symmetric about each standard hyperplane $x_i = 0$, assume without loss of generality that $(x, y, z, w) \geq 0$, then if we show that $(x, y, z, w) \in Q$, the reflections of this point will also be.

Then $\langle (x, y, z, w), (1, 1, 0, 0) \rangle \leq 1$ and $\langle (x, y, z, w), (0, 0, 1, 1) \rangle \leq 1$. Hence $x + y + z + w \leq 2$ Abbreviate $m = \frac{x+y+z+w}{4}$ Then it follows that

$$(x - m)e_1 + (y - m)e_2 + (z - m)e_3 + (w - m)e_4 + 2m\frac{e_1 + e_2 + e_3 + e_4}{2} = (x, y, z, w)$$

is a convex combination of the generators of Q, so that $(x, y, z, w) \in Q$. Hence $Q = P^{\text{o}}$

Finally we exhibit an invertible linear transformation T from Q onto P. I will simply write it in terms of the standard basis, but by brute force, we can show that T is a bijection from the generating set of Q to the generating set of P, so that it will induce a map on the convex hulls themselves.

Let $T(1, 0, 0, 0) = (1, 1, 0, 0)$, $T(0, 1, 0, 0) = (1, -1, 0, 0)$, $T(0, 0, 1, 0) = (0, 0, 1, 1)$ and $T(0, 0, -1, 1)$.
This transformation is invertible and suffices. □

Chapter 7: Lattices and Convex Bodies

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3. Let $\Lambda \subset \mathbb{R}^d$ be a lattice. Then $\Lambda \cap B(x_0, r)$ is a finite set.

   Proof. Assume $x_0 \in \mathbb{R}^d$ and $r > 0$.

   Since Λ is discrete in $\mathbb{R}^d$, there exists $\epsilon > 0$ such that $B(\lambda, \epsilon) \cap \Lambda = \{\lambda\}$ for every $\lambda \in \Lambda$. Furthermore, since Λ is discrete, every point Λ is isolated, meaning it contains all of its limit points, so Λ is a closed subset of $\mathbb{R}^d$. Let $B = \overline{B(x_0, r)}$.

   Since B is a closed and bounded subset of $\mathbb{R}^d$, it follows that B is compact, and from its definition we have $B(x_0, r) \subseteq B$. Since both B and Λ are closed subsets of $\mathbb{R}^d$, we see that

$B \cap \Lambda$ is as well. Furthermore, since B is compact and $B \cap \Lambda$ is closed in B, we see that $B \cap \Lambda$ is compact in $\mathbb{R}^d$.

With all of this, we see that since $\{B(\lambda, \epsilon)\}_{\lambda \in \Lambda}$ is an open cover of $B \cap \Lambda$, there exists $\lambda_1, ..., \lambda_n \in \Lambda$ such that $B \cap \Lambda \subset \bigcup_{k=1}^n B(\lambda_k, \epsilon)$. Furthermore, from the discreteness of Λ, we see that for each k the only element of Λ in $B(\lambda_k, \epsilon)$ is λ_k itself. Therefore $B \cap \Lambda = (B \cap \Lambda) \cap \Lambda \subseteq (\bigcup_{k=1}^n B(\lambda_k, \epsilon)) \cap \Lambda = \{\lambda_1, ..., \lambda_n\}$.

Therefore we see that $\Lambda \cap B(x_0, r) \subseteq \Lambda \cap B \subseteq \{\lambda_1, ..., \lambda_n\}$, which since the set on the right is a finite set, it follows that $\Lambda \cap B(x_0, r)$ is finite as well. $\square$

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We have seen the following examples of lattices:

(a) $\mathbb{Z}^d$

(b) $A_d = \mathbb{Z}^{d+1} \cap \{(\xi_1, ..., \xi_{d+1}) \in \mathbb{R}^{d+1} : \xi_1 + ... + \xi_{d+1} = 0\}$.

(c) $D_n = \{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \xi \in \mathbb{Z}$ and $\xi_1 + ... + \xi_{d+1}$ is even$\}$

(d) $D_n^+ = D_n \cup (D_n + x_0)$, where $x_0 = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \in \mathbb{R}^n$ is a lattice for n even, not for n odd. D_8^+ is called E_8.

(e) $E_7 = E_8 \cap \{(\xi_1, ..., \xi_8) : \xi_1 + \xi_2 + ... + \xi_8 = 0\}$

(f) $E_6 = E_8 \cap \{(\xi_1, ..., \xi_8) : \xi_1 + \xi_8 = 0$ and $\xi_2 + \xi_3 + ... + \xi_7 = 0\}$

1. Find bases for A_n, D_n, D_n^+, E_6, and E_7.

Proof. Basis for A_n:
For each $k \in \{1, ..., n\}$, $u_k = (1, 0, ..., 0, -1, 0, ..., 0)$, where -1 occurs at the $k+1$-th spot.

Basis for D_n:
$u_1 = (2, 0, ..., 0)$, $u_k = (1, 0, ..., 0, 1, 0, ..., 0)$, where 1 occurs at the kth spot for $k \in \{2, ..., n\}$.

Basis for D_n^+:
$u_1 = (2, 0, ..., 0)$, $u_2 = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, and for $k \in \{2, ..., n-1\}$, $u_k = (0, ..., 0, -1, 1, 0, ..., 0)$, where -1 occurs at the $(k-1)$-th spot and 1 occurs at the k-th spot.

Basis for E_7:

$$\begin{aligned} u_1 &= (0, -1, 1, 0, 0, 0, 0, 0), & u_5 &= (0, 0, 0, 0, 0, -1, 1, 0), \\ u_2 &= (0, 0, -1, 1, 0, 0, 0, 0), & u_6 &= (0, 0, 0, 0, 0, 0, -1, 1), \\ u_3 &= (0, 0, 0, -1, 1, 0, 0, 0, 0), & u_7 &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), \\ u_4 &= (0, 0, 0, 0, -1, 1, 0, 0) \end{aligned}$$

Basis for E_6:

$$\begin{aligned} u_1 &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}) & u_4 &= (0, 0, 0, -1, 1, 0, 0, 0) \\ u_2 &= (0, -1, 1, 0, 0, 0, 0, 0) & u_5 &= (0, 0, 0, 0, -1, 1, 0, 0) \\ u_3 &= (0, 0, -1, 1, 0, 0, 0, 0) & u_6 &= (0, 0, 0, 0, 0, -1, 1, 0) \end{aligned}$$

$\square$

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1-3. Find the determinant of A_n, D_n, D_n^+, E_6, and E_7.

Proof. Using the bases found above, we compute the determinant:

$\det(A_n) = \sqrt{n+1}$

$\det(D_n) = 2$

$\det(D_n^+) = 1$

$\det(E_6) = \sqrt{3}$

$\det(E_7) = \sqrt{2}$

□

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7. Prove that two linearly independent vectors $u, v \in \mathbb{Z}^2$ are a basis of $\mathbb{Z}^2$ if and only if the triangle with vertices 0, u, v contains no other integer points other than 0, u, and v.

Proof. We will prove that $\{u, v\}$ are a basis for $\mathbb{Z}^2$ by proving the contrapositive statement. To that end, assume there exists $z \in \mathbb{Z}^2$ in the triangle with vertices 0, u, and v that is not equal to 0, u, or v.

Note that z being in this triangle implies $z = au + bv + c0 = au + bv$, where $0 \leq a, b, c \leq 1$ and $a + b + c = 1$. Note that since $z \notin \{0, u, v\}$, it cannot be that $a, b, c \in \{0, 1\}$. Therefore $0 < a, b < 1$, forcing $a, b \notin \mathbb{Z}$. This means that $z = au + bv$, where $a, b \notin \mathbb{Z}$, meaning $\{u, v\}$ cannot be a basis for $\mathbb{Z}^2$.

Now assume that the triangle with vertices 0, u, and v contains no other integer points other than 0, u, and v. Note that the triangle with vertices 0, u, and v is equal to $Conv\{0, u, v\}$. Write $T = Conv\{0, u, v\}$.

Note that since $T \cap \mathbb{Z}^2 = \{0, u, v\}$, it follows that $-T \cap \mathbb{Z}^2 = \{0, -u, -v\}$, since $w \in -T \cap \mathbb{Z}^2$ implies $-w \in T \cap \mathbb{Z}^2$, meaning $-w \in \{0, u, v\}$. Also note that if $w \in T \cap \mathbb{Z}^2$, then w is an integer combination of u and v.

Assume $z \in \mathbb{Z}^2$. Then z is either an integer combination of u or v or it is not. If it is, then $z \in Span\{u, v\}$ (as a lattice). We'll show that this is the only option for z.

To that end, suppose the contrary and suppose there exists $z \in \mathbb{Z}^2$ which is not equal to $au+bv$ for some $a, b \in \mathbb{Z}$. Then since u and v are linearly independent in $\mathbb{Z}^2$, they span $\mathbb{R}^2$, so there exists $s, t \in \mathbb{R}$ such that $z = su + tv$. Then there exists $m, n \in \mathbb{Z}$ such that $m \leq s < m + 1$ and $n \leq t < n + 1$. Then we see that either $z \in Conv\{mu + nv, (m + 1)u, (n + 1)v\}$ or $z \in Conv\{(m + 1)u, (n + 1)v, (m + 1)u + (n + 1)v\}$.

For the first case, we may translate $Conv\{mu+nv, (m+1)u, (n+1)v\}$ by the vector $(-mu) + (-nv)$, so that $Conv\{mu+nv, (m+1)u, (n+1)v\} + (-mu) + (-nv) = Conv\{0, u, v\}$, so that $z - mu - nv \in \{0, u, v\}$, meaning z is an integer combination of u and v, which contradicts the assumption.

Similarly, for the second case, if $z \in Conv\{(m+1)u, (n+1)v, (m+1)u+(n+1)v\}$, then translating the set $Conv\{(m+1)u, (n+1)v, (m+1)u+(n+1)v\}$ by $-(m+1)u-(n+1)v$ shows us that $z-(m+1)u-(n+1)v \in Conv\{0, -u, -v\}$, which in turn means that $(m+1)u+(n+1)v-z \in Conv\{0, u, v\}$, again meaning z is an integer combination of u and v, which again is a contradiction.

Therefore no z can both be in $\mathbb{Z}^2$ and not be an integer combination of u and v. Therefore, as lattices, $\mathbb{Z}^2 = Span\{u, v\}$, which means that u and v are a basis for the lattice $\mathbb{Z}^2$. □

8. Give an example of linearly independent vectors $u, v, w \in \mathbb{Z}^3$ such that $conv(0, u, v, w)$ contains no integer points in its interior, but $\{u, v, w\}$ is not a basis of $\mathbb{Z}^3$.

   Let $u = (1,0,0)$, $v = (0,1,0)$, and $w = (1,1,2)$. Then u, v, and w are linearly independent vectors in $\mathbb{Z}^3$, but $(1,1,1)$ is not in the span of u, v, and w. We'll show that $Conv\{0, u, v, w\} \cap \mathbb{Z}^3 = \{0, u, v, w\}$, which will prove the claim.

   To that end, assume $z \in Conv\{0, u, v, w\} \cap \mathbb{Z}$. Then $z = au + bv + cw + d0 = au + bv + cw$, where $a, b, c, d > 0$ and $a + b + c + d = 1$.

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5. Two lattices Λ_1, Λ_2 in $\mathbb{R}^d$ are similar ($\Lambda_1 \sim \Lambda_2$) if one is obtained from the other by a composition of an orthogonal transformation of $\mathbb{R}^d$ and a dilation. Show that $D_2 \sim \mathbb{Z}^2$, $D_3 \sim A_3$, and $D_4^+ \sim \mathbb{Z}^4$.

   Proof. $\underline{D_2 \sim \mathbb{Z}^2}$

   Let
$$A = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$
   and let $B = \frac{1}{\sqrt{2}}A$. Then since $AA^T = Id_2$, A is an orthogonal transformation of $\mathbb{R}^2$, and so B is a dilation of A. Using the basis $\{[2,0]^T, [1,1]^T\}$ of D_2, we see
$$B\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
$$B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
   Therefore B is a dilation of an orthogonal transformation which takes a basis of the lattice D_2 to a basis of the lattice $\mathbb{Z}^2$, meaning B is the required transformation. Therefore $D_2 \sim \mathbb{Z}^2$.

   $\underline{D_3 \sim A_3}$

   Take for a basis of D_3, the collection $\{(1,1,0), (1,0,1), (0,1,1)\}$ and for A_3, the collection $\{(1,-1,0,0), (1,0,-1,0), (1,0,0,-1)\}$
   Then there is a unique linear transformation mapping the first basis onto the subspace spanned

by the second basis; we compute the change of basis matrix $T = A^{-1}B$ where A is the matrix with rows given by the basis for D_3 and B is the matrix with rows given by the basis for A_3:

$$T = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

This transformation maps the Lattice D_3 to A_3(via multiplication on the left), and the rows of the transformation are orthonormal. So it follows that D_3 can be obtained from A_3 by such a transformation.

$\underline{D_4^+ \sim \mathbb{Z}^4}$ Using Wolfram Alpha, we found that an orthonormal basis for D_4^+ is $\{f_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), f_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), f_3 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), f_4 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. With this, define a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

With this matrix, we see that $TT^T = Id_4$, so T is an orthogonal matrix. Furthermore, we see that if $\{e_n\}_{n=1}^4$, then $T(e_n) = f_n$. Therefore T moves $\mathbb{Z}^4$ into D_4^+, meaning $T(\mathbb{Z}^4) = D_4^+$, so T is the desired matrix. Therefore $D_4^+ \sim \mathbb{Z}^4$.

□

6. Find the packing radii of $\mathbb{Z}^d$, A_n and D_n for $n \geq 2$, D_2^+, D_4^+, D_6^+, D_n^+ for $n \geq 8$, E_6, and E_7.

Proof. Packing Radius for $\mathbb{Z}^d$: $\frac{1}{2}$ (coming from the vector $(1, 0, ..., 0)$)

Packing Radius for A_n: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Packing Radius for D_n: $n \geq 2$: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ (for $n = 2$, $(1, 1)$ is shorter than $(2, 0)$, so it has the smallest norm. For $n > 2$ even, $(-1, 1, 0, ..., 0)$ is tied for or is of shortest norm, so it is chosen)

Packing Radius for D_2^+: $\frac{1}{2\sqrt{2}}$

Packing Radius for D_4^+: $\frac{1}{2}$

Packing Radius for D_6^+: $\frac{\sqrt{6}}{4}$

Packing Radius for D_n^+, $n \geq 8$: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ (the vector $(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$ has norm greater than $\sqrt{2}$ when $n \geq 8$, so the vector $(-1, 1, 0, ..., 0)$ will always be smaller in D_n^+).

Packing Radius for E_6: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Packing Radius for E_7: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

□

7. Compute the packing density of A_2, A_3, D_4, D_5, E_6, E_7, and E_8.

Proof. Remember $\sigma(\Lambda) = vol_d(B(0,\rho))\frac{1}{\det \Lambda}$.
Packing Density of A_2: $\sigma(A_2) = vol_2(B(0, \frac{1}{\sqrt{2}}))\frac{1}{\det A_2} = \pi(\frac{1}{\sqrt{2}})^2 \cdot \frac{1}{\sqrt{3}} = \frac{\pi}{2\sqrt{3}}$

Packing Density of A_3: $\sigma(A_3) = vol_3(B(0, \frac{1}{\sqrt{2}}))\frac{1}{\det A_3} = \frac{4}{3}\pi(\frac{1}{\sqrt{2}})^3 \frac{1}{\sqrt{4}} = \frac{\pi}{3\sqrt{2}}$.

Packing Density of D_4: $\sigma(D_4) = vol_4(B(0, \frac{1}{\sqrt{2}}))\frac{1}{\det D_4} = \frac{\pi^2}{2}(\frac{1}{\sqrt{2}})^4 \cdot \frac{1}{2} = \frac{\pi^2}{16}$

Packing Density of D_5: $\sigma(D_5) = vol_5(B(0, \frac{1}{\sqrt{2}}))\frac{1}{\det D_5} = \frac{8\pi^2}{15}(\frac{1}{\sqrt{2}})^5 \frac{1}{2} = \frac{\pi^2}{15\sqrt{2}}$

Packing Density of E_6: $\sigma(E_6) = vol_6(B(0, \frac{1}{\sqrt{2}}))\frac{1}{\det E_6} = \frac{\pi^3}{6}(\frac{1}{\sqrt{2}})^6 \frac{1}{\sqrt{3}} = \frac{\pi^3}{48\sqrt{3}}$

Packing Density of E_7: $\sigma(E_7) = vol_7(B(0, \frac{1}{\sqrt{2}}))\frac{1}{\det E_7} = \frac{16}{105}\pi^3(\frac{1}{\sqrt{2}})^7 \frac{1}{\sqrt{2}} = \frac{16}{1680}\pi^3$

Packing Density of E_8: $\sigma(E_8) = vol_8(B(0, \frac{1}{\sqrt{2}}))\frac{1}{\det D_8^+} = \frac{\pi^4}{24}(\frac{1}{\sqrt{2}})^8 \frac{1}{1} = \frac{\pi^4}{2^4 4!} = \frac{\pi^4}{384}$

□

Problems about the Euler Formula

1. Use the Euler formula to show that there are exactly 5 regular polytopes in $\mathbb{R}^3$.

Proof. Let P be a regular polytope. Since P is regular, we see that every edge has the same length, and every angle measure on all faces are the same. This forces the shape of each face to all be a regular polygon (and the same one as each other at that). Therefore, if m represents the number of edges on a single face F of P, then it actually represents the number of edges on any face.

Now, project P onto a sphere surrounding it. Looking at a vertex p on one of the projected faces, we see that the sum of the angles will be 2π, and there will be n edges meeting at p, and the angle will have measure $\frac{2\pi}{n}$. For another vertex q of any face, we see that the number of angles meeting at q will be the same as the number of angles meeting at p. Since the angle measures will have to be the same, it follows that the measure of each angle meeting at q will also be $\frac{2\pi}{n}$. Since the total angle measure has to be 2π, it follows that there would have to be n angles and n edges. Therefore the number of edges meeting at a point is the same as at every other point.

Now, since each edge has the same number of edges meeting at it, and since each edge meets exactly two vertices, we see that $2e = nv$. Note also that each edge lies on the border of exactly 2 faces, and since each face has m edges on it, we see that $2e = mf$. With this, we find that $v = \frac{2e}{n}$ and $f = \frac{2e}{m}$. Therefore by Euler's formula we see

$$2 + e = v + f = \frac{2e}{n} + \frac{2e}{m}.$$

On the other side, we also see

$$\frac{2e}{n} + \frac{2e}{m} = 2 + e > e,$$

which implies

$$\frac{1}{n} + \frac{1}{m} > \frac{1}{2}.$$

Remember that a face of a polygon must have at least 3 edges. Furthermore, if only two edges met at a point, then the shape would have to be a single face, but that violates Euler's Characteristic Formula (for example, it might have $(v, e, f) = (3, 3, 1)$, but $v + f = 3 + 1$ and $2 + e = 2 + 3 = 5$ are unequal). Therefore $m, n \geq 2$.

Now, we plug in different natural numbers for m and n to find that the following pairs of (m, n) of natural numbers satisfy the above inequality:

$$(3, 3), (3, 4), (4, 3), (3, 5), (5, 3).$$

Existence of these can be seen by looking at a set of specialty dice (i.e. for Dungeons and Dragons) from a gaming store. Uniqueness follows from Cauchy's Rigidity Theorem. Therefore there are 5 polytopes which satisfy these conditions, each one corresponding to a unique pair of numbers from above. □

2. Use Euler's formula ($f_0 - f_1 + f_2 - f_3 = 0$) to determine the regular 4-polytopes.

Proof. Aside: We can determine the regular 4-polytopes using dihedral angles of the platonic solids. The facets of a regular 4-polytope must be regular 3-polytopes. The number of facets meeting at an edge must be at least 3 (similar to how at least three faces meet at a vertex for 3-polytopes). Additionally, the sum of the dihedral angles of the facets cannot exceed 360°. Therefore, using the approximated dihedral angles below, we can determine the possibilities for the regular 4-polytopes.

Notation: let m be the number of edges per face of a regular 3-polytope, and let n be the number of faces meeting at a vertex. We can identify each shape by the pair $\{m, n\}$.

{3,3}	Tetrahedron	70.53°
{4,3}	Cube	90°
{3,4}	Octahedron	109.47°
{5,3}	Dodecahedron	116.53°
{3,5}	Icosahedron	138.19°

We see that we can fit 3, 4, or 5 tetrahedra around an edge, but not 6. Similarly we can fit 3 cubes, octahedra, and dodecahedra around an edge, but not 4. Finally, we cannot even fit 3 icosahedra around an edge. Let p be the number of facets meeting at an edge. Then the regular 4-polytopes can be identified by the triple $\{m, n, p\}$ where m determines the 2-d faces, and $\{m, n\}$ determines the 3-d facets as above. Thus the six regular 4-polytopes are the shapes determined by $\{3, 3, 3\}$ (simplex), $\{3, 3, 4\}$, $\{3, 3, 5\}$, $\{4, 3, 3\}$ (hypercube or tesseract), $\{3, 4, 3\}$, and $\{5, 3, 3\}$.

Preliminary work: Let m, n, and p be defined as above. Let

$$\phi(m, n) = \text{number of faces on each 3-d facet} = \frac{4n}{4 - (m-2)(n-2)}$$

. Define $\mathcal{E}$ to be the number of edges meeting at each vertex (for the entire 4-polytope). Now, we can write Euler's formula in terms of f_1. We get the relation $\mathcal{E}f_0 = 2f_1$ since each edge is connected to two vertices, the relation $mf_2 = pf_1$ since p faces meet at each edge, and the relation $\phi(m,n)f_3 = 2f_2$ since we double counted the faces. Now, we get the following rewritten Euler's formula:

$$f_1\left(\frac{2}{\mathcal{E}} - 1 + \frac{p}{m} - \frac{2p}{m\phi(m,n)}\right) = 0$$

. Since we know that the facets have to be regular 3-polytopes, we can obtain restrictions on p and $\mathcal{E}$ by considering each 3-polytope separately and using the appropriate values of m and n. For instance, regular 4-polytopes with cubic faces ($\{4,3\}$) gives the equation

$$48 - 24\mathcal{E} + 4p\mathcal{E} = 0$$

Now, since p has to be greater than or equal to 3 and $\mathcal{E}$ has to be a natural number, we get bounds on p. In this case $p = 3, 4, 5$. Unfortunately, this method results in 11 possible regular 4-polytopes (note that for $\{4,3\}$ we should only get $p = 3$ for the 4-polytope $\{4,3,3\}$).

$\square$