falls between the texts on Euclidean or projective geometry currently available. Borsuk's book [85] is possibly the most comprehensive text for this purpose, but its notation is quite outdated.

If $A$ is an $n \times n$ matrix, the inverse and transpose of $A$ are denoted by $A^{-1}$ and $A^t$. We call $A$ *singular* or *nonsingular* according to whether $\det A = 0$ or $\det A \neq 0$, respectively; $A^{-1}$ exists precisely when $A$ is nonsingular. We also adopt the abbreviation $A^{-t}$ for $(A^{-1})^t$. Note that if $A$ is nonsingular, then $A^t$ is also, and $(A^t)^{-1} = (A^{-1})^t$.

For transformations $\phi$ of $\mathbb{E}^n$ and $\mathbb{P}^n$, we shall permit ourselves the shorthand $\phi x = \phi(x)$. The reader may find Figure 0.1 useful in interpreting the definitions given below.

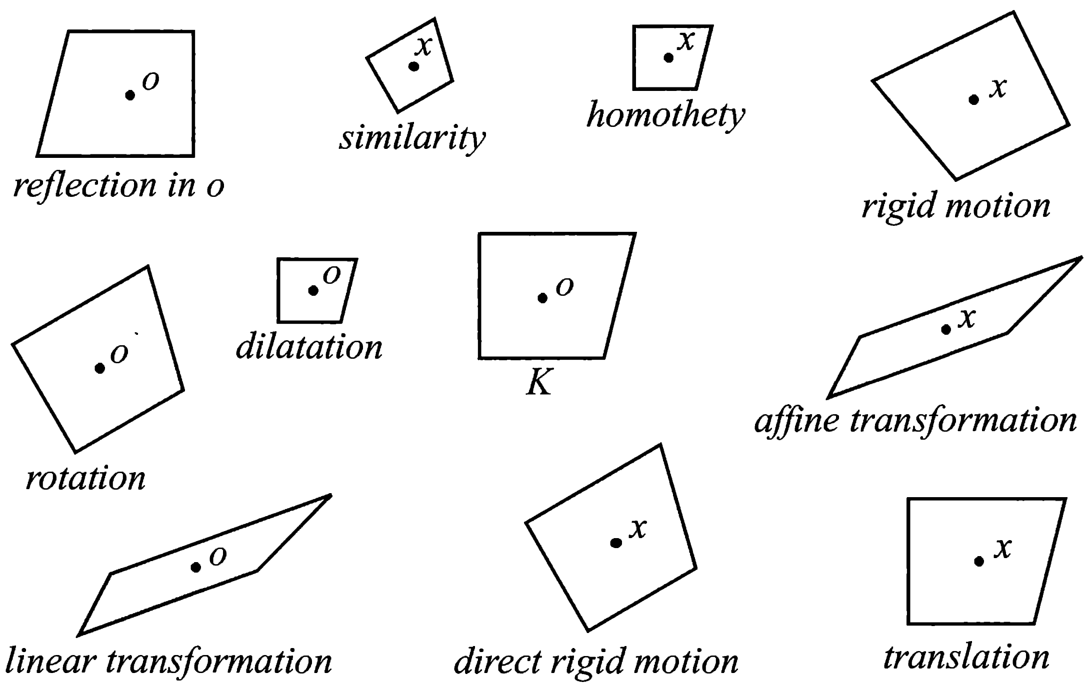


Figure 0.1. Transformations of a set $K$.

A *linear transformation* (or *affine transformation*) of $\mathbb{E}^n$ is a map $\phi$ from $\mathbb{E}^n$ to itself such that $\phi x = Ax$ (or $\phi x = Ax + t$, respectively), where $A$ is an $n \times n$ matrix and $t \in \mathbb{E}^n$. (Here $x$ is considered as a column vector, of course.) We call $\phi$ *singular* or *nonsingular* according to whether $A$ is singular or nonsingular, respectively. The group of nonsingular linear (or affine) transformations is denoted by $GL_n$ (or $GA_n$); its members are, in particular, bijections of $\mathbb{E}^n$ onto itself. The group of *special linear* (or *special affine*) transformations of $\mathbb{E}^n$ is denoted by $SL_n$ (or $SA_n$, respectively). These are the members of $GL_n$ (or $GA_n$) whose determinant is one. We shall write $\det \phi$ instead of $\det A$, and $\phi^{-1}$, $\phi^t$, and $\phi^{-t}$ for the affine transformations with corresponding matrices $A^{-1}$, $A^t$, and $A^{-t}$, respectively.

If $A$ is the identity matrix, then $\phi x = x + t$, and the map $\phi$ is called a *translation*. Each affine transformation is composed of a linear transformation followed by a translation.

Any set of $n + 1$ points in general position in $\mathbb{E}^n$ can be mapped onto any second set of $n + 1$ points by a suitable affine transformation, and the latter is nonsingular if the second set is also in general position (see [595, Theorem 7, p. 16]).

If $\phi \in GA_n$, then $\phi$ takes parallel $k$-dimensional planes onto parallel $k$-dimensional planes (cf. [85, p. 156]).

An *isometry* of $\mathbb{E}^n$ is a map $\phi$ such that $\|\phi x - \phi y\| = \|x - y\|$; in other words, a distance-preserving bijection. Isometries are also called congruences, and the image and pre-image under an isometry are said to be *congruent*. Every isometry is affine (see, for example, [85, p. 150] or [839, p. 139]). Examples of isometries are the translations and the *reflections*, which map all points to their mirror images in some fixed point, line, or plane. (In particular, $\phi x = -x$ is the reflection in the origin.)

If $F = S + x_0$ (where $S \in \mathcal{G}(n,k)$, $x_0 \in \mathbb{E}^n$, and $1 \le k \le n-1$) is a $k$-dimensional plane, and $x \in \mathbb{E}^n$, then there are unique points $y \in S$ and $z \in S^\perp$ such that $x = y + z$, and we can define a map taking $x$ to $y + x_0 \in F$. This map is the (orthogonal) *projection* on the plane $F$. It is a singular affine transformation. If $E$ is an arbitrary subset of $\mathbb{E}^n$, the image of $E$ under a projection on a plane $F$ is called the *projection of $E$ on $F$* and denoted by $E|F$. Since $E|S$ is a translate of $E|F$ when $F = S + x_0$, we almost always work with the former.

If $\phi \in GL_n$, then

$$x \cdot \phi y = \phi^t x \cdot y, \tag{0.1}$$

for all $x, y \in \mathbb{E}^n$. The *orthogonal group $O_n$* of orthogonal transformations consists of those isometries of $\mathbb{E}^n$ that are also linear transformations; these are precisely the maps $\phi$ preserving the scalar product, that is, $\phi x \cdot \phi y = x \cdot y$. (An orthogonal matrix satisfies $A^t = A^{-1}$ and by (0.1) we have $\phi^t = \phi^{-1}$, hence the name.) It follows from this that orthogonal transformations have determinants with absolute value one. As is shown in [85, Theorem 50.6], every isometry is an orthogonal transformation followed by a translation, and for this reason isometries are sometimes also called *rigid motions*. The *special orthogonal group $SO_n$* of *rotations* about the origin consists of those orthogonal transformations with determinant one. A *direct rigid motion* is a rotation followed by a translation; these do not allow reflection.

A *dilatation* is a map $\phi x = rx$, for some $r > 0$. A *homothety* is a map $\phi x = rx + t$, for some $r > 0$ and $t \in \mathbb{E}^n$, that is, a composition of a dilatation with a translation (this is sometimes referred to as a direct homothety). A *similarity* is a composition of a dilatation with a rigid motion. We say two sets are *homothetic* (or *similar*) if one of them is an image of the other under a homothety (or similarity, respectively), or if one of the sets is a single point.

We find occasional use for projective transformations of $\mathbb{P}^n$. Such a transformation is given in terms of homogeneous coordinates by $\phi w = Aw + t$, where $A$ is an $(n+1) \times (n+1)$ matrix and $t \in \mathbb{E}^{n+1}$, and where $\phi$ is called nonsingular if $\det A \neq 0$. Since we can regard $\mathbb{P}^n$ as $\mathbb{E}^n$ with a hyperplane $H_\infty$ adjoined, we can also speak of a projective transformation of $\mathbb{E}^n$. In this regard, another formulation is useful. A *projective transformation* $\phi$ of $\mathbb{E}^n$ has the form

$$\phi x = \frac{\psi x}{x \cdot y + t}, \tag{0.2}$$

where $\psi \in GA_n$, $y \in \mathbb{E}^n$, and $t \in \mathbb{R}$, and $\phi$ is nonsingular if the associated linear map

$$\bar{\psi}(x, 1) = (\psi x, x \cdot y + t)$$

is nonsingular. If $y = o$, then $\phi$ is affine, but if $y \neq o$, $\phi$ maps the hyperplane $H = \{x : x \cdot y + t = 0\}$ onto $H_\infty$. To avoid points in a set $E$ being mapped into $H_\infty$, we may insist that $\phi$ be *permissible* for $E$; this simply means that $E \cap H = \emptyset$.

Projective transformations map planes onto planes (neglecting the points mapping to or from infinity); see [595, pp. 19–20]. They also preserve cross ratio; a proof is given in [85, Corollary 96.11]. (The *cross ratio* of four points $x_i$, $1 \leq i \leq 4$ on a line is defined by

$$\langle x_1, \ldots, x_4 \rangle = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_4 - x_1)(x_3 - x_2)},$$

where $x_i$ also denotes the coordinate of the point $x_i$ in a fixed Cartesian coordinate system on the line.) Affine transformations are also projective transformations, so the former also preserve cross ratio.

The sets $E$ and $F$ are called *linearly, affinely*, or *projectively equivalent* if there is a nonsingular transformation $\phi$, linear, affine, or projective and permissible for $E$, respectively, such that $\phi E = F$. Suppose that $E$ and $F$ are bounded centered sets affinely equivalent via a nonsingular transformation $\phi$. If $\phi o = p$, then $p$ is the center of $F$; but since $o$ is the unique center of $F$, we have $p = o$. Therefore $\phi$ is linear, proving that $E$ and $F$ are linearly equivalent.

## 0.3. Basic convexity

There are several possibilities for an introduction to the basic properties of convex sets. For the absolute beginner, the books of Lay [499] and Webster [827] are recommended. The first chapter of [595], by McMullen and Shephard, is terse, but very informative, as is the first chapter of [737], by Schneider. The text of [845], by Yaglom and Boltyanskiĭ, is set out in the form of exercises and solutions, with