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AN EQUICHORDAL PROBLEM

ABSTRACT. Two points C and C' in the interior of a convex domain in the plane have the property that each chord of K through C' has the same length as the parallel chord through C. Then K is necessarily centrally symmetric about the mid-point of CC'.

1. INTRODUCTION

A point E in the interior of a convex domain K in the plane is said to be an equichordal point, if each chord of K through E has the same length. The infamous equichordal problem is whether or not a convex domain in the plane can have two distinct equichordal points. Despite the efforts of many mathematicians, and despite the remarkable work of E. Wirsing (1958) and of G.J. Butler (1969) proving the existence of convex domains with a pair of points C,C' that are equichordal for all chords making a sufficiently small angle with the line CC', this problem appears to be most intractable.

Recently P. McMullen asked me about quite a different equichordal problem. He starts with the remark that, if C,C' are inner points of a convex domain K that is centrally symmetric about the mid-point of the line segment CC', then each chord of K through C has the same length as the parallel chord of K through C'. Here, of course, the lengths of the chords will depend on the angle that they make with the line CC'. McMullen asks for a proof of the converse.

THEOREM 1. *Let C,C' be distinct inner points of a convex domain K. If each chord of K through C has the same length as the parallel chord of K through C', then K is centrally symmetric about the mid-point of the line segment CC'.*

The object of this note is to provide a simple proof of this result. D.G. Larman and N. Tamvakis have obtained an n-dimensional generalization.

2. TWO LEMMAS

In this section we prove two lemmas. The first lemma does not make use of the equichordal property. It will be convenient to fix some notation before we state the lemma. C and C' are to be two distinct points in the interior of the convex domain K (but we do not yet take C and C' to be an equichordal pair). Let the points V and V' be the points of intersection of the line through C and C' with the boundary of K, and suppose that V and V' are named so

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that V', C', C and V lie in this order on the chord $V'V$ of K. Let (x, y) be coordinates chosen so that

$$C' = (-1, 0), \qquad C = (1, 0).$$

With each point P_0 of the boundary of K, in the half-plane $y > 0$, we associate two sequences $P_0, P_1, P_2, \ldots$ and $P'_0, P'_1, P'_2, \ldots$ on the boundary of K, the first sequence in the half-plane $y > 0$, and the second sequence in the half-plane $y < 0$. When P_r has been defined, for $r \geqslant 0$, we define P'_r to be the point, on the boundary of K in the half-plane $y < 0$, on the line through C' parallel to the line segment CP_r. When P'_r has been defined, for $r \geqslant 0$, in the half-plane $y < 0$, we define P_{r+1} to be the second intersection, necessarily in the half-plane $y > 0$, of the line P'_rC with the boundary of K. This clearly provides an inductive definition of the required sequences $P_0, P_1, P_2, \ldots$ and $P'_0, P'_1, P'_2, \ldots$ associated with P_0.

LEMMA 1.

$$P_r \to V \quad \textit{and} \quad P'_r \to V' \qquad \textit{as} \quad r \to \infty.$$

Proof. For each $r \geqslant 0$, the lines $C'P'_r$ and P_rC are parallel, and the angles $C\hat{P}'_rC'$ and $P_{r+1}\hat{C}P_r$ are equal. Hence

$$V\hat{C}P_{r+1} = V\hat{C}P_r - P_{r+1}\hat{C}P_r = V\hat{C}P_r - C\hat{P}'_rC'.$$

Furthermore, as C lies between C' and V, the angle $CP'_r\, C'$ is strictly positive and $V\hat{C}P_{r+1}$ is strictly less than $V\hat{C}P_r$. If the sequence of angles $V\hat{C}P_r$, $r = 0, 1, 2, \ldots$ were to converge to a positive limit, the points P_r, $r = 0, 1, 2, \ldots$ would converge to a point Q on the boundary of K in the half-plane $y > 0$, and the point P'_r would converge to the point Q', on the boundary of K in the half-plane $y < 0$, on the line through C' parallel to the line CQ. Now $Q'CQ$ is not a straight line, but for each $r \geqslant 0$, P'_rCP_{r+1} is a straight line and

$$P'_r \to Q', \qquad P_{r+1} \to Q$$

as $r \to \infty$. This contradiction shows that $V\hat{C}P_r$ converges to zero as $r \to \infty$. Thus $P_r \to V$ and $P'_r \to V'$ as $r \to \infty$.

From now on we suppose that the points C and C' form an equichordal pair, i.e., that each chord of K through C' has the same length as the parallel chord of K through C.

LEMMA 2.

$$V'C' = CV.$$

Proof. We suppose that $V'C' \neq CV$ and obtain a contradiction. We suppose, as we may without loss of generality, that $V'C' > CV$. Consider a point P

that moves continuously from V' to V round that part of the boundary of K lying in the half-plane $y \geqslant 0$. Let P' be the point, on the boundary of K in the half-plane $y \leqslant 0$, on the line through C' parallel to CP. Then P' moves continuously from V to V' along the boundary of K in the half-plane $y \leqslant 0$ as P moves from V' to V. When P is at V', we have

$$C'P' = C'V = C'C + CV < C'C + V'C' = CV' = CP.$$

When P is at V, we have

$$C'P' = C'V' > CV = CP.$$

By continuity, for some suitable P_0, on the boundary of K in the half-plane $y > 0$, we have

$$C'P_0' = CP_0.$$

Let $P_0, P_1, P_2, \ldots$ and $P_0', P_1', P_2', \ldots$ be the sequences associated with this choice of P_0. Suppose that, for some $r \geqslant 0$, we have

$$(1) \qquad C'P_r' = CP_r.$$

Then $C'P_r'$ is equal and parallel to CP_r. Hence $CP_rC'P_r'$ is a parallelogram and CP_r' is equal and parallel to $C'P_r$. But $P_r'CP_{r+1}$ is a chord of K. So $P_{r+1}'C'P_r$ is the parallel chord of K, through C', to $P_r'CP_{r+1}$. As C and C' form an equichordal pair,

$$P_r'CP_{r+1} = P_{r+1}'C'P_r.$$

Subtracting the common value $P_r'C = C'P_r$ we obtain

$$C'P_{r+1}' = CP_{r+1}.$$

As (1) holds when $r = 0$, it follows, by induction, that (1) holds for all $r \geqslant 0$. Letting $r \to \infty$ in (1) and using Lemma 1, we obtain

$$C'V' = CV,$$

as required.

3. Proof of Theorem 1

We suppose that our coordinates are chosen as in the last section. As the situation is invariant under any affine transformation, we may suppose further that the half-tangent to K, at the point V lying in the half-plane $y > 0$, is parallel to the y-axis. Consider any point $P_0 = (x_0, y_0)$ on the boundary of K, with

$$1 < x_0, \qquad 0 < y_0.$$

Suppose further that x_0 is so large that the slopes of the half-tangents to K at any point (x, y) with

$$x \geqslant x_0, \qquad y > 0,$$

are negative, perhaps having the value $-\infty$. Let $P_0, P_1, P_2, \ldots$ and $P'_0, P'_1, P'_2, \ldots$ be the sequences of points on the boundary of K associated with the point P_0. Let P_r and P'_r have coordinates (x_r, y_r) and (x'_r, y'_r) for $r \geqslant 0$.

We prove that $x_0 + x'_0 = 0$. To do this, we suppose that $x_0 + x'_0 \neq 0$ and seek a contradiction. We use induction to prove that, if $x_0 + x'_0 > 0$, then

$$(2) \qquad x_r + x'_r \geqslant x_0 + x'_0, \quad \text{for} \quad r \geqslant 0,$$

and if $x_0 + x'_0 < 0$, then

$$(3) \qquad x_r + x'_r \leqslant x_0 + x'_0, \quad \text{for} \quad r \geqslant 0.$$

Suppose that for some value $r \geqslant 0$ we either know that

$$x_r + x'_r \geqslant x_0 + x'_0 > 0,$$

or that

$$x_r + x'_r \leqslant x_0 + x'_0 < 0.$$

Let the line $P'_{r+1}C'$ meet the boundary of K at a second point Q_{r+1} with coordinates (ξ_{r+1}, η_{r+1}), and let this line $P'_{r+1}C'$ meet the line CP_r at a point R_{r+1}. Then $CR_{r+1}C'P'_r$ is a parallelogram. If

$$x_r + x'_r > 0,$$

then CP_r is longer than P'_rC', and R_{r+1} is an inner point of the line segment CP_r, so that Q_{r+1} lies on $C'R_{r+1}$ produced beyond R_{r+1}. As the slopes of the half-tangents to K at P_r are negative, it follows that

$$\xi_{r+1} \geqslant x_r.$$

On the other hand, if

$$x_r + x'_r < 0,$$

then CP_r is shorter than P'_rC', and R_{r+1} is on CP_r produced beyond P_r, so so that Q_{r+1} is an inner point of the line segment $C'R_{r+1}$. As the slopes of the half-tangents to K at P_r are negative, it follows, in this case, that

$$\xi_{r+1} \leqslant x_r.$$

As the chords $P'_{r+1}C'Q_{r+1}$ and P'_rCP_{r+1} are parallel, they have equal lengths, by the equichordal property. Hence

$$\xi_{r+1} - x'_{r+1} = x_{r+1} - x'_r,$$

or

$$x_{r+1} + x'_{r+1} = \xi_{r+1} + x'_r.$$

In the first case, when $x_0 + x_0' > 0$, we have

$$x_{r+1} + x_{r+1}' \geqslant x_r + x_r' \geqslant x_0 + x_0',$$

and in the second case, when $x_0 + x_0' < 0$, we have

$$x_{r+1} + x_{r+1}' \leqslant x_r + x_r' \leqslant x_0 + x_0'.$$

Thus, by induction, we either have $x_0 + x_0' > 0$ and (2) holds for all $r \geqslant 0$, or we have $x_0 + x_0' < 0$ and (3) holds for all $r \geqslant 0$.

By Lemmas 1 and 2 we see that

$$x_r + x_r' \to 0$$

as $r \to \infty$. In either of the two cases, this gives a contradiction. Hence we must have $x_0 + x_0' = 0$. Thus P_0' is the reflection of P_0 in the origin. As the argument applies to any point P on the arc $P_0 V$ of the boundary of K, in the half-plane $y \geqslant 0$ joining P_0 to V, it follows that the arc $P_0' V'$ of the boundary of K, in the half-plane $y < 0$ joining P_0' to V', is the reflection of the arc $P_0 V$ in the origin.

Now consider any point S_0 on the boundary of K in the half-plane $y > 0$. Let $S_0, S_1, S_2, \ldots,$ and $S_0', S_1', S_2', \ldots$ be the sequences associated with S_0. By Lemma 1, we have

$$S_r \to V, \qquad S_r' \to V',$$

as $r \to \infty$. Provided r is sufficiently large, we have

$$S_r \in P_0 V \quad \text{and} \quad S_r' \in P_0' V'.$$

Hence S_r' is the reflection of S_r in the origin, and

$$C'S_r' = CS_r,$$

for all sufficiently large r. Using the argument of Lemma 2 backwards, we obtain

$$C'S_r' = CS_r, \quad \text{for all} \quad r \geqslant 0.$$

Hence $C'S_0' = CS_0$, and S_0' is the reflection of S_0 in the origin. Thus K is itself centrally symmetric in the mid-point of the segment CC'.

BIBLIOGRAPHY

1. Wirsing, E. 'Zur Analytizität von Doppelspeichenkurven'. *Arch. Math.*, **9** (1958), 300–307.

2. Butler, G.J. 'On the "equichordal curve" problem and a problem of packing and covering'. Thesis (London, 1969).
3. Larman D.G. and Tamvakis, N. K. "A characterisation of centrally symmetric convex bodies in E^n." to appear in *Geometriae Dedicata.*

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