MATH 756
HW1: Due Monday Jan 28

1. Let $p \in (1, \infty)$ and $f \in L^p[0,1]$. Show that

$$\lim_{y \to 0^+} y^{\frac{1-p}{p}} \int_0^y f(x)\, dx = 0.$$

2. Let (X, μ) and (Y, ν) be σ-finite measure spaces and let $K \in L^2(\mu \times \nu)$. For $f \in L^2(\nu)$, we define the operator

$$Tf(x) = \int_Y K(x,y) f(y) d\nu(y).$$

Prove

(a) $Tf(x)$ is well defined for μ-a.e. x.

(b) $Tf \in L^2(\mu)$.

(c) $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$.

3. For this problem, read first Theorem 6.20 and Corollary 6.21 (and their proofs) in Folland.

(a) Let $K(x)$ be a non-negative measurable function on $(0, \infty)$ such that for every $0 < s < 1$,

$$\int_0^\infty K(x) x^{s-1} dx = \phi(s) < \infty.$$

Let $1 < p < \infty$, let q be its conjugate exponent, and let f, g be non-negative measurable functions on $(0, \infty)$. Show

$$\int_0^\infty \int_0^\infty K(xy) f(x) g(y) dx dy \leq \phi(p^{-1}) \left[\int_0^\infty x^{p-2} f(x)^p dx \right]^{1/p} \left[\int_0^\infty g(x)^q dx \right]^{1/q}.$$

(b) Use (a) to show that the operator

$$Tf(x) = \int_0^\infty K(xy) f(x) dx$$

is bounded on $L^2(0, \infty)$ with norm less than or equal $\phi(1/2)$.

(c) What are T and ϕ in the particular case where $K(x) = e^{-x}$?

(d) Let $1 < p < \infty$ and let q be its conjugate exponent. Define

$$Tf(x) = x^{-1/p} \int_0^x f(t)dt.$$

Show that T is a bounded linear map from $L^q(0, \infty)$ to $C_0(0, \infty)$ (the set of continuous functions vanishing at infinity).

4. Show that weak L^p is a vector space.

5. Let (X, μ) be a measure space.

(a) Let $1 \leq p < r \leq \infty$, and let $f \in L^p(\mu) \cap L^r(\mu)$. Prove that $f \in L^q(\mu)$ for every $p < q < r$.

(b) Let $0 < p < q < \infty$. Show that $L^p(\mu) \not\subset L^q(\mu)$ if and only if X contains sets of arbitrarily small positive measure.

(c) Let $0 < p < q < \infty$. Show that $L^q(\mu) \not\subset L^p(\mu)$ if and only if X contains sets of arbitrarily large finite measure.

(d) Prove that if f is in weak L^p and $\mu(\{x : f(x) \neq 0\} < \infty$, then $f \in L^q$ for every $q < p$. On the other hand, if f is in weak L^p and in L^∞, then $f \in L^q$ for all $q > p$.