MATH 757: Homework 2

1. (a) Let M be a positive real number. Show that there exists $\phi \in L^1(\mathbb{R})$ such that $\phi \star f = f$ for every $f \in L^2(\mathbb{R})$ such that $supp(\widehat{f}) \subset [-M, M]$.

   (b) Prove that there is no $\phi \in L^1(\mathbb{R})$ such that $\phi \star f = f$ for every $f \in L^2(\mathbb{R})$.

2. Show the following:

   (a) If $f(x) = \frac{1}{1+x^2}$, then $f \star f \star f = \frac{3\pi^2}{9+x^2}$.

   (b) If $|x| < \frac{|A|}{2\pi}$, then $\displaystyle\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(Ay)}{y} e^{2\pi i x y}\, dy = \pi$.

   (c) If $f(x) = \frac{e^{10\pi i x}}{9+x^2}$, then $\widehat{f}(\xi) = \frac{\pi}{3} e^{-6\pi|\xi-5|}$.

3. This problem presents an alternate proof of the known fact that C_c^∞ is dense in L^p, for $p < \infty$. Let $f \in L^p(\mathbb{R})$ for some p, $1 \le p < \infty$.

   (a) Show that if $\phi \in \mathcal{S}$, then $\phi \star f \in C^\infty \cap L^p$.

   (b) Assume that $\int \phi = 1$ and define $\phi_n(x) = n\phi(nx)$. Then $\{\phi_n\}$ is a family of good kernels, and we have seen in class that $\|\phi_n \star f - f\|_p \to 0$ as $n \to \infty$. Consider another function $\psi \in \mathcal{S}$ such that $\chi_{[-1,1]} \le \psi \le \chi_{[-2,2]}$, and let $\psi^n(x) = \psi(x/n)$. Prove:

      - $(\phi_n \star f) \cdot \psi^n \in C_c^\infty$.
      - $\|(\phi_n \star f) \cdot \psi^n - f\|_p \to 0$ as $n \to \infty$.

4. The goal of this problem is to check that the Fourier transform is well-defined on $L^2(\mathbb{R})$. We defined in class the Fourier transform of $f \in L^2(\mathbb{R})$ as the function $G \in L^2$ such that $\lim_{n\to\infty} \|G - \widehat{f_n}\|_2 = 0$, where $\{f_n\}$ is a sequence of Schwartz functions satisfying $\lim_{n\to\infty} \|f - f_n\|_2 = 0$.

   (a) Given the above G and another sequence $\{g_n\}$ of Schwartz functions satisfying $\lim_{n\to\infty} \|f - g_n\|_2 = 0$, show that we also have $\lim_{n\to\infty} \|G - \widehat{g_n}\|_2 = 0$. Hence G does not depend on the chosen sequence.

   (b) For a function $f \in L^1(\mathbb{R})$, we have the original definition $\widehat{f}(\xi) = \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$. Use the approximating functions in problem 3 to show that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, both definitions of the Fourier transform agree.

5. **Eigenfunctions for the Fourier transform operator**. In this exercise we will study functions f such that $\widehat{f} = \lambda f$. We have seen in class that the function $e^{-\pi x^2}$ is equal to its Fourier transform, so it is an eigenfunction associated to the eigenvalue 1.

(a) Using a result seen in class, show that the only possible eigenvalues are $1, -1, i, -i$.

(b) For each integer $n \geq 0$, the Hermite functions are defined by $h_n(x) = (-1)^n e^{x^2/2} \left(\frac{d}{dx}\right)^n e^{-x^2}$. Show that $h_n \in \mathcal{S}$ for each $n \geq 0$. Find the explicit expressions for h_0 and h_1.

(c) Note that each Hermite function is of the form $h_n(x) = p_n(x)e^{-x^2/2}$, where p_n is a polynomial of degree n. The Hermite polynomial can clearly be defined by $p_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$ (*i.e.*, we just kill the exponential term in the definition of h_n). Hermite polynomials are known to satisfy several recursions (you do not need to prove (1)-(3)).

$$p_n'(x) = 2xp_n(x) - p_{n+1}(x) \tag{1}$$

$$p_n'(x) = 2np_{n-1}(x) \tag{2}$$

$$p_{n+1}(x) = 2xp_n(x) - 2np_{n-1}(x) \tag{3}$$

Show first that $\displaystyle\int_{-\infty}^{\infty} h_0(x)h_1(x)dx = 0$, and then use the above recursion formulas for the polynomials to prove that $\{h_n\}_{n\geq 0}$ is an orthogonal family in $L^2(\mathbb{R})$.

(d) Use Taylor series to show that

$$\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} = e^{-(x^2/2-2tx+t^2)}.$$

Hint: Write $e^{-(x^2/2-2tx+t^2)} = e^{x^2/2}e^{-(x-t)^2}$ and expand the term depending on t.

(e) Show that if $f \in L^2(\mathbb{R})$ and $\displaystyle\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\, dy = 0$ for every $x \in \mathbb{R}$, then $f \equiv 0$ a.e. Hint: Consider $f \star e^{-x^2}$.

(f) Combine (d) and (e) to show that if $f \in L^2$ and $\displaystyle\int_{-\infty}^{\infty} f(y)h_k(y)\, dy = 0$ for every $k \geq 0$, then $f \equiv 0$. Hence the family $\{h_n\}_{n\geq 0}$ is an orthogonal basis in L^2.

(g) Let $H_n(x) = h_n(\sqrt{2\pi}x)$. Show that $\widehat{H_n}(\xi) = (-i)^n H_n(\xi)$. Hence the H_n are the eigenfunctions of the Fourier transform operator. (Hint: Find the FT of H_0, H_1, H_2 by hand. For the general case, use the Taylor series formula in (d), take Fourier transforms on both sides. You can use Mathematica for the Fourier transform of the right hand side).

(h) **Conclusion:** We have proven that the family $\left\{\dfrac{H_n}{\|H_n\|}\right\}_{n\geq 0}$ is an orthonormal basis for $L^2(\mathbb{R})$ consisting on (all) the eigenfunctions for the Fourier transform. This means that $L^2(\mathbb{R})$ decomposes into the direct sum $H_0 \oplus H_1 \oplus H_2 \oplus H_3$, where on each subspace H_j the Fourier transform acts by multiplying functions by $(-i)^j$. This approach to defining the Fourier transform in L^2 is due to N. Wiener. See Wikipedia for lots of information and references about these functions and their recursions.

6. **Multipliers for the Fourier transform in $L^2(\mathbb{R})$.** Given a measurable function m, we define the operator T by the formula $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$. Prove:

(a) T is linear.

(b) T maps $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ if and only if $m \in L^\infty(\mathbb{R})$.

(c) If $m \in L^\infty(\mathbb{R})$, the operator norm of $T : L^2 \to L^2$ equals $\|m\|_\infty$.