MATH 756
HW2: Due Friday Feb 15

1. Show that the triangle inequality fails for weak L^p spaces: Find two functions f, g in $L^p[0,1]$ (with Lebesgue measure) such that $[f]_p = 1$, $[g]_p = 1$ and $[f+g]_p > 2$.

2. Let f^* be the decreasing rearrangement of f (see Folland, Exercise 40 on pag 199). Prove that f, f^* have equal distribution functions. (We saw in class the inequality $\lambda_{f^*}(\alpha) \leq \lambda_f(\alpha)$).

3. Write out in detail the proof of Marcinkiewicz Interpolation Theorem in the cases

(a) $p_0 = q_0 = 1, p_1 = q_1 = 2$

(b) $p_0 = q_0 = 1, p_1 = q_1 = \infty$

4. Let f be a continuous function on $[0, +\infty)$ (with Lebesgue measure). For $\alpha > 0$ and $x \geq 0$, define the **fractional integral operator**

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt.$$

(a) Show that for $x, y > 0$,

$$\Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

(b) Show that $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$ (use (a)).

(c) Show that if $n \in \mathbb{N}$, $I_n f$ is an n-th order antiderivative of f.

(d) If $\alpha < 1$ and $1 < p < \alpha^{-1}$, show that I_α is of weak type $(1, (1-\alpha)^{-1})$.

(e) With p as in (d), let $r^{-1} = p^{-1} - \alpha$. Show that I_α is of strong type (p, r).