MATH 756
HW3: Due Friday March 1

1. Let $F_N(x)$ be the N-th Fejer kernel, *i.e.* $\dfrac{1}{N+1}\displaystyle\sum_{k=0}^{N} D_k(x)$, where $\{D_k(x)\}$ are the Dirichlet kernels.

(a) Show that the Fejer kernel can also be written as follows:

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\right)^2.$$

(b) Compute the Fourier coefficients of F_N.

(c) Show that for every $j \in \mathbb{Z}$, $\lim_{N\to\infty} \widehat{F_N}(j) = 1$.

(d) Prove that $\lim_{N\to\infty} \|F_N\|_2 = \infty$.

(e) Compute $\lim_{N\to\infty} \dfrac{\|F_N\|_2}{\|D_N\|_2}$.

2. Let $f \in L^1(\mathbb{T})$ such that $\{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Show that there is a countinuous function g such that $g = f$ a.e.

3. Riemann-Lebesgue Lemma says that the Fourier coefficients of a function in $L^1(\mathbb{T})$ tend to zero. This exercise shows that they can tend arbitrarily slowly. Given a sequence of positive numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$, prove that there is a function $f \in C[0,1]$ such that $|\widehat{f}(n)| + |\widehat{f}(-n)| \geq \epsilon_n$ for infinite values of n (hint: consider an appropriate subsequence of $\{\epsilon_n\}$ and use Problem 2).

4. Compute the Fourier coefficients of the function $f(x) = ax + bx^2 + cx^3$, defined on $[0,1]$ and extended periodically to $\mathbb{R}$. Use the coefficients to find the numerical value of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2},\ \sum_{n=1}^{\infty}\frac{1}{n^4},\ \sum_{n=1}^{\infty}\frac{1}{n^6}.$$

Turn over

5. Compute the Fourier series of the following functions (considered defined on [0,1) and extended periodically).

(a) $f_1(x) = \sin(4\pi x)\cos(2\pi x)$

(b) $f_2(x) = \sin(x)$

(c) $f_3(x) = e^x$

(d) $f_4(x) = \cosh(2\pi x)$

(e) $f_5(x) = \chi_(1/3, 2/3)(x)$