MATH 857 SPRING 2023
HW 1 - Due Feb 15

1. **A Calculus uniqueness problem:**

   Let $r(s) = \langle x(s), y(s), z(s)\rangle$ be a non-planar curve in $\mathbb{R}^3$, parametrized by the arclength. Let L be a fixed line in $\mathbb{R}^3$. Prove that the following are equivalent:

   (a) All tangent lines to $r(s)$ make a constant angle with L

   (b) All normal lines to $r(s)$ are perpendicular to L

   (c) All binormal lines to $r(s)$ make a constant angle with L.

   (d) The ratio $k(s)/\tau(s)$ between the curvature and the torsion is constant for s.

   What is the curve characterized by any one of these properties?

   Note: All the necessary definitions (tangent, normal, binormal, curvature and torsion) are in any Calculus III textbook, there is no need to look for a more advanced text.

2. **A useful result for convex sets:** Let A, B be convex bodies in $\mathbb{R}^n$ such that $A \subseteq B$ and $\partial A \subseteq \partial B$. Show that $A = B$. Indicate with a counterexample where convexity, compactness and/or non-empty interior are needed.

3. **Convex functions:** Let $f : (a,b) \to \mathbb{R}$ be a convex function (recall that the definition of convex function is that for every $x, y \in \mathbb{R}$ and $t \in (0,1)$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$).

   (a) Prove that f is convex if and only if for every $x, y, u, v \in (a,b)$ such that $x \leq u < v$ and $x < y \leq v$,
   $$\frac{f(y) - f(x)}{y - x} \leq \frac{f(v) - f(u)}{v - u}.$$

   (b) Prove that f is convex if and only if f is absolutely continuous on every compact subinterval of (a,b) (which implies that f' exists a.e.) and f' is increasing on the set where it is defined.

   (c) Show that if f is convex and $t_0 \in (a,b)$, there exists $\beta \in \mathbb{R}$ such that $f(t) - f(t_0) \geq \beta(t - t_0)$ for every $t \in (a,b)$.

   (d) **Jensen's inequality:** Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$, let $g : X \to (a,b)$ be in $L^1(\mu)$, and f be convex on (a,b). Then,
   $$f\left(\int_X g d\mu\right) \leq \int_X f \circ g d\mu.$$
   Use (c), setting $t = g(x)$ and $t_0 = \int_X g d\mu$.

4. **A characterization of the Euclidean ball:** Let K be a convex body in $\mathbb{R}^n$.

   (a) Show that K is a ball if and only if for every pair of different points $x, y \in \partial K$, there exists $b \in \mathbb{R}^n$ and $T \in O(n)$ such that $T(K) + b = K$ and $T(x) + b = y$.

   (b) Use (a) to prove that if a convex body has two different axes of revolution, it must be a ball.