MATH 857 SPRING 2023
HW 2 - Due March 8

1. Let γ be a regular curve in $\mathbb{R}^3$.
   (a) Assuming that γ is twice differentiable, prove that if all tangent lines to γ pass through a fixed point P, then γ is a line.
   (b) Prove the same result assuming only that γ is differentiable.
   (c) If γ is twice differentiable and all normal lines pass through a fixed point, then γ is contained on a sphere.
   (d) If γ is twice differentiable and all the osculating planes pass through a fixed point, then γ is a plane curve.

2. Let $r(s) = (x(s), y(s))$ be the arc length parametrization of a regular plane curve γ, and $\phi(x, y) = 0$ the implicit equation of another curve C. Let P be a common point to both curves. The curves γ and C have order of contact n at $P = r(s_0)$ iff the following equations are satisfied:

$$\begin{aligned} \phi(x(s_0), y(s_0)) &= 0 \\ \frac{d}{ds}\phi(x(s_0), y(s_0)) &= 0 \\ &\vdots \\ \frac{d^n}{ds^n}\phi(x(s_0), y(s_0)) &= 0. \end{aligned}$$

   Similarly, if $r(s) = (x(s), y(s), z(s))$ is the parametrization of curve in $\mathbb{R}^3$ and $\phi(x, y, z) = 0$ is the implicit equation of a surface, the contact of order n is defined by the same system of equations, with the additional variable $z(t)$.

   (a) Prove that if the curvature $k_1(s_0) \neq 0$, then the *osculating circle* of γ at P (i.e. the circle with contact of order 2) has center in the normal direction at P and radius $1/k_1(s_0)$.
   (b) Prove that the equation of the *osculating sphere* to a curve γ at P (i.e. the sphere with contact of order 3) has radius $\sqrt{(\frac{1}{k_1})^2 + (\frac{k_1'}{k_1^2 k_2})^2}$ and center $P + \mathbf{n}/k1 + \mathbf{b}(\frac{k_1'}{k_1^2 k_2})$, where $\mathbf{n}, \mathbf{b}$ are the normal and binormal vectors of γ at P.

More problems on the back $\rightarrow$

3. A *diameter* of a convex body K is a chord of maximal length (maximal length is attained since K is compact). A convex body K may have more than one diameter (e.g. the ball).

   A *double normal* of K is chord $[a, b]$ of K, with $a, b \in \partial K$, such that the normal vectors of K at the points a and b have the same direction as $[a, b]$.

   For a strictly convex body K, a *diametral chord* of K is a chord joining two boundary points of K whose support hyperplanes are parallel.

   Prove the following facts:

   (a) A chord $[a, b]$ is a diametral chord of K iff it is the longest chord of K in the direction of the segment $[a, b]$.

   (b) For every $x \in K$ there is a diametral chord of K containing x.

   (c) Every diameter is a double normal.

   (d) An ellipsoid in $\mathbb{R}^n$ has n double normals.

4. Let K be a convex body in $\mathbb{R}^n$. Show that the following conditions are equivalent:

   (a) For every direction $u \in S^{n-1}$ there is a double normal of K parallel to u.

   (b) Any point $x \in K$ lies on a double normal of K.

   (c) Every double normal of K is a diameter of K.

   (d) K has constant width.

5. Prove that if K is strictly convex, then K is centrally symmetric with center x_0 if and only iff all diametral chords of K pass through x_0.

   Hint: For the converse, you may assume that K is 2-dimensional (why?). Let x_0 be the origin, and consider a parametrization $r(t)$ of ∂K and the parametrization $r(t + \pi)$ of the boundary of the reflection of K through x_0. Study the condition on parallel support lines simultaneously for both figures in terms of $r(t)$.