MATH 857 SPRING 2023
HW 3 - Due March 29

1. **Loxodromes:** In class, we saw how to compute the family of curves that are orthogonal to the family $x =$constant on the hyperbolic paraboloid $z = axy$. Use a similar method to compute the loxodromes on a sphere (curves that form a constant angle α with the meridians, also called). Plot some of the curves for different values of α. On the Mercator projection of the earth, loxodromes become straight lines, which made them very useful for navigation, as the pilot only needed to keep a constant compass direction.

2. **A property of paraboloids:** Consider the circular paraboloid $z = \frac{a}{2}(x^2 + y^2)$ and the hyperbolic paraboloid $z = axy$. Prove that for any domain D on the xy-plane, the surface area of the region above D is equal for both paraboloids. (Hint: Compute the First Fundamental Forms).

3. **Surfaces of Revolution:** Let γ be a curve on the xz-plane, parametrized by $x = \phi(u), z = \psi(u)$. A surface of revolution obtained by rotating the curve γ around the z-axis can be parametrized by

$$r(u, v) = (\phi(u)\cos v, \phi(u)\sin v, \psi(u).$$

The lines $v =$constant are called the meridians of the surface of revolution, and the lines $u =$constant are called the parallels.

   (a) Compute the First Fundamental Form of a surface of revolution (in terms of ϕ, ψ). What can we conclude about the meridians and parallels?

   (b) Compute the Second Fundamental Form of a surface of revolution. Use it to determine which are the lines of curvature.

   (c) The *catenoid* is the surface of revolution given by

$$r_C(u, v) = (a\cosh u \cos v, a\cosh u \sin v, au), \quad u \in (0, 2\pi), v \in \mathbb{R}. \tag{1}$$

   The helicoid has the usual parametrization

$$r_H(u, v) = (\widetilde{u}\cos \widetilde{v}, \widetilde{u}\sin \widetilde{v}, a\widetilde{v}), \quad \widetilde{u} \in (0, 2\pi), \widetilde{v} \in \mathbb{R}. \tag{2}$$

   Make the change of variables $\widetilde{u} = a\sinh u, \widetilde{v} = v$ on the helicoid and show that it is locally isometric to the catenoid by computing their First Fundamental Forms. This implies, in particular, that the Gauss curvatures at corresponding points of the catenoid and the helicoid are equal, which is a nontrivial result.

4. Determine the asymptotic curves of the catenoid (1) and the helicoid (2). Find also the lines of curvature of the helicoid (2).

5. A parametric surface is called *minimal* if its mean curvature is zero at all points. These are the surfaces obtained by dipping a wire in a soap solution. Show that the catenoid is a minimal surface (in fact, it can be shown that it is the only minimal surface of revolution). What can we conclude about the helicoid?