ONESIDED APPROXIMATION BY ENTIRE FUNCTIONS

FRIEDRICH LITTMANN

Abstract. Let $f : \mathbb{R} \to \mathbb{R}$ have an nth derivative of finite variation $V_{f^{(n)}}$ and a locally absolutely continuous $(n-1)$st derivative. Denote by $E^{\pm}(f,\delta)_p$ the error of onesided approximation of f (from above and below, respectively) by entire functions of exponential type $\delta > 0$ in $L^p(\mathbb{R})$–norm. For $1 \le p \le \infty$ we show the estimate

$$E^{\pm}(f,\delta)_p \le C_n^{1-1/p} \pi^{1/p} V_{f^{(n)}} \delta^{-n-\frac{1}{p}},$$

with constants $C_n > 0$.

1. Introduction

This article considers the problem of onesided approximation of real-valued functions defined on the real line by entire functions of finite exponential type $\delta > 0$.

As part of his investigation of Beurling's extremal majorant of $\operatorname{sgn}(x)$ and related functions, J. D. Vaaler showed in Theorem 11 of [6] that onesided approximation in $L^1(\mathbb{R})$ by functions of exponential type δ is possible with an error bounded by $\pi V_f \delta^{-1}$. (V_f denotes the total variation of f on $\mathbb{R}$.) D. Dryanov [2] proved a similar result with an error given in terms of an integrated modulus of continuity under the additional restriction that $f \in L^1(\mathbb{R})$.

Below, Vaaler's result is generalized to functions f having the property that $f^{(n-1)}$ is absolutely continuous and $V_{f^{(n)}}$ is finite. Let $\mathcal{A}(\delta)$ be the class of entire functions of exponential type δ, i.e., $F \in \mathcal{A}(\delta)$ if and only if for every $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that

$$|F(z)| \le A_\varepsilon e^{(\delta+\varepsilon)|z|} \text{ for all } z \in \mathbb{C}.$$

Throughout the paper it is assumed that f is real-valued and normalized in the sense that $f(x) = 2^{-1}(f(x+) + f(x-))$ holds.

Definition 1. Let f be a real-valued function whose $(n-1)$st derivative is locally absolutely continuous. The error function $E^+(\delta, f)_p$ is defined as the infimum of $||A_+ - f||_p$ taken over all $A^+ \in \mathcal{A}(\delta)$ which satisfy $A_+ \ge f$ on the real line. ($E^-(\delta, f)_p$ is defined using the reverse inequality.)

In previous investigations, the error of onesided approximation has been given in terms of $||A^+ - A^-||_p$. However, in anticipation of the statements of Lemma 1 and Lemma 5, it is preferable to keep upper and lower approximation separate.

Theorem 1. *If f is real valued, has an nth derivative with bounded variation, and (for $n \ge 1$) a locally absolutely continuous $(n-1)$st derivative, then*

$$E^{\pm}(\delta, f)_p \le C_n^{1-\frac{1}{p}} \pi^{\frac{1}{p}} V_{f^{(n)}} \cdot \delta^{-n-\frac{1}{p}}, \tag{1}$$

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where $1 \le p \le \infty$*, and* $C_n > 0$ *is some constant.*

The proof of Theorem 1 follows the approach of [6]. There, the function f is written as an integral convolution of f with the Dirac measure $1/2\,d(\mathrm{sgn})$ and is approximated by replacing $\mathrm{sgn}(x)$ with its best (lower and upper) approximations of exponential type $\delta > 0$.

Here we use the same idea, i.e., f is approximated by a convolution of $f^{(n)}$ with measures $dG_n^\pm$ where the functions $G_n^\pm$ are the best upper and lower approximation of type δ to the truncated powers x_+^n, respectively. The approximations $G_n^\pm$ were obtained in [3]. We review their properties in the next section.

It is worth pointing out that, since $\mathbb{R}$ is only locally compact, the class of functions having an nth derivative of bounded variation is not contained in the class of functions having an $(n-1)$st derivative of bounded variation, i.e., the statement of Theorem 1 for $n+1$ is not a refinement of the statement for n.

2. Extremal Functions for the Truncated Powers

Let $x_+ = x$ for $x \ge 0$ and $x_+ = 0$ for $x < 0$. We require the following facts about onesided best approximation of x_+^n.

Let $\alpha \in \mathbb{R}$ and define

$$\mathscr{G}_{n,\alpha}(z) := \frac{\sin^2 \pi(z-\alpha)}{\pi^2} z^n \Big[\psi'(\alpha - z) + \sum_{j=0}^{n} B_j(\alpha) z^{-j-1}\Big], \tag{2}$$

where $\psi = \Gamma'/\Gamma$ with the Euler Gamma function Γ, and B_j denotes the jth Bernoulli polynomial.

For even $n > 0$ we define z_n to be the zero of the nth Bernoulli polynomial in $(0, 1/2)$, and we set $z_0 := 0$. We define for $n \in \mathbb{N}_0$

$$\alpha_n := \begin{cases} 1 - z_n, \\ 0, \\ z_n, \\ 1/2, \end{cases} \qquad \beta_n := \begin{cases} z_n & \text{if } n \equiv 0 \mod 4, \\ 1/2 & \text{if } n \equiv 1 \mod 4, \\ 1 - z_n & \text{if } n \equiv 2 \mod 4, \\ 0 & \text{if } n \equiv 3 \mod 4. \end{cases} \tag{3}$$

These values are the maxima and minima on $[0,1]$ of the $(n+1)$st Bernoulli polynomial.

Lemma 1 (cf. Theorem 7.2 of [3])**.** *Let* $n \in \mathbb{N}_0$ *and* $\delta > 0$*. The functions* $\delta^{-n}\mathscr{G}_{n,\alpha_n}(\delta x)$ *and* $\delta^{-n}\mathscr{G}_{n,\alpha_n}(\delta x)$ *are the unique best onesided approximations from* $\mathcal{A}(2\pi\delta)$ *to* x_+^n*. In particular, they satisfy* $\delta^{-n}\mathscr{G}_{n,\alpha_n}(\delta x) \le x_+^n \le \delta^{-n}\mathscr{G}_{n,\beta_n}(\delta x)$ *for all* $x \in \mathbb{R}$ *and*

$$\int_{\infty}^{\infty} \left(\delta^{-n}\mathscr{G}_{n,\beta_n}(\delta x) - x_+^n\right) dx = -\frac{B_{n+1}(\beta_n)}{n+1}\delta^{-n-1}, \tag{4}$$

$$\int_{\infty}^{\infty} \left(x_+^n - \delta^{-n}\mathscr{G}_{n,\alpha_n}(\delta x)\right) dx = \frac{B_{n+1}(\alpha_n)}{n+1}\delta^{-n-1}. \tag{5}$$

For the proof of Lemma 1 see [3].

We define for real x

$$\psi_{n,\alpha}(x) := \mathscr{G}_{n,\alpha}(x) - x_+^n. \tag{6}$$

The following lemma expresses symmetry properties of $\psi_{n,\alpha}$.

Lemma 2. *Let $n \in \mathbb{N}_0$. If n is even, then*

$$\psi_{n,\alpha_n}(x) = -\psi_{n,\beta_n}(-x),$$

if n is odd, then

$$\begin{aligned}\psi_{n,\alpha_n}(x) &= \psi_{n,\alpha_n}(-x),\\ \psi_{n,\beta_n}(x) &= \psi_{n,\beta_n}(-x).\end{aligned}$$

Proof. Let F_n and G_n be best onesided approximations from $\mathcal{A}(2\pi)$ to $\mathrm{sgn}(x)x^n$ with $F_n(x) \geq \mathrm{sgn}(x)x^n \geq G_n(x)$ for $x \in \mathbb{R}$. Since $2x_+^n$ and $\mathrm{sgn}(x)x^n$ differ by a polynomial (which is of exponential type zero), we have

$$\begin{aligned}\psi_{n,\beta_n}(x) &= 2^{-1}(F_n(x) - \mathrm{sgn}(x)x^n),\\ \psi_{n,\alpha_n}(x) &= 2^{-1}(G_n(x) - \mathrm{sgn}(x)x^n).\end{aligned} \tag{7}$$

Since best onesided $L^1(\mathbb{R})$–approximations from $\mathcal{A}(2\pi)$ to x_+^n are unique (cf. Theorem 7.2 of [3]), the functions F_n and G_n are unique best onesided approximations to $\mathrm{sgn}(x)x^n$. Since $(-1)^{n+1}F_n(-x)$ and $(-1)^{n+1}G_n(-x)$ are also (best) onesided $L^1(\mathbb{R})$–approximations to $\mathrm{sgn}(x)x^n$, we obtain $F_n(x) = F_n(-x)$ and $G_n(x) = G_n(-x)$ for odd n and $F_n(x) = -G_n(-x)$ for even n, since otherwise F_n and G_n would not be unique best approximations. Lemma 2 follows now from (7).

Lemma 3. *If $n \in \mathbb{N}_0$ and $\alpha \in \{\alpha_n, \beta_n\}$ then the derivatives $[\mathscr{G}_{n,\alpha}(x) - x_+^n]^{(k)}$ for $0 \leq k \leq n$ and $\mathscr{G}_{n,\alpha}^{(n+1)}$ are absolutely integrable on the real line.*

Proof. Let $\alpha \in \{\alpha_n, \beta_n\}$. By equations (4.4) and (4.6) in [3], $\mathscr{G}_{n,\alpha}$ has the representations

$$\begin{aligned}\mathscr{G}_{n,\alpha}(x) &= \frac{F(x-\alpha)}{x}\int_{-\infty}^{0} e^{-xt}\gamma_\alpha^{(n+1)}(t)dt \qquad \text{for } x<0,\\ &= x^n - \frac{F(x-\alpha)}{x}\int_{0}^{\infty} e^{-xt}\gamma_\alpha^{(n+1)}(t)dt \qquad \text{for } x>0,\end{aligned}$$

where $F(x) = \pi^{-2}\sin^2 \pi x$ and $\gamma_\alpha(t) = te^{\alpha t}(e^t - 1)^{-1}$.

Let $k \in \{1, ..., n+1\}$. We obtain after k differentiations with respect to x and an integration by parts with respect to t that if $k \leq n$, then $(\mathscr{G}_{n,\alpha}(x) - x_+^n)^{(k)} \leq C_1|x|^{-2}$, and if $k = n+1$, then $\mathscr{G}_{n,\alpha}^{(n+1)}(x) \leq C_2|x|^{-2}$ as $|x| \to \infty$.

Remark 1. If $F - x_+^n \in L^1(\mathbb{R})$ for an entire function F of finite exponential type, then $(F - x_+^n)^{(k)}$ for $k \leq n$ and $F^{(n+\ell)}$ for $\ell \geq 1$ are also elements of $L^1(\mathbb{R})$. This can be seen by adding and subtracting $\mathscr{G}_{n,\beta_n}$, and then applying a theorem of Plancherel and Polya [5] to $F - \mathscr{G}_{n,\beta_n}$ and Lemma 3 to $\mathscr{G}_{n,\beta_n} - x_+^n$.

3. Entire approximation

We define the function x_-^n to be x^n for $x < 0$ and 0 for $x > 0$.

Lemma 4. *If $f^{(n-1)}$ is locally absolutely continuous and $f(0) = f'(0) = ... = f^{(n-1)}(0) = 0$, then we can represent $f(x)$ in the form*

$$f(x) = \begin{cases} -(n!)^{-1} \displaystyle\int_0^\infty f^{(n)}(u)d[(x-u)_+^n] & \textit{for } x \geq 0,\\ (n!)^{-1} \displaystyle\int_{-\infty}^0 f^{(n)}(u)d[(x-u)_-^n] & \textit{for } x \leq 0.\end{cases}$$

Proof. For $n = 0$, the claim follows from the fact that $d(x_+^0)$ is the Dirac measure at the origin.

Let n be a positive integer and assume $x > 0$. Under the assumption of the lemma, we have

$$-\int_0^\infty f^{(n)}(u)d(x-u)_+^n = n\int_0^x f^{(n)}(u)(x-u)^{n-1}du,$$

and the claim follows for $n = 1$ directly and for $n > 1$ with an integration by parts and induction on n. The computations for $x < 0$ can be done in the same way.

Let $V(x, f)$ be the total variation of f on the interval $(-\infty, x)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with a locally absolutely continuous $(n-1)$st derivative, and with an nth derivative which has finite variation. By a classical theorem of real analysis (Theorem VII of §516 in [4]), the functions $f_n^\pm(x) := 2^{-1}(f^{(n)}(x) \pm V(x, f^{(n)}))$ are non-decreasing, have bounded variation, satisfy

$$f^{(n)} = f_n^+ - f_n^- \tag{8}$$

and the sum of their total variations equals the total variation of $f^{(n)}$.

We establish now the main result of this paper.

Lemma 5. *Let f be real-valued with $V_{f^{(n)}} < \infty$ and let $\delta > 0$. There are entire functions $A_{f,n,\delta}^+$ and $A_{f,n,\delta}^-$ of exponential type at most $2\pi\delta$ which satisfy the conditions*

$$A_{f,n,\delta}^-(x) \le f(x) \le A_{f,n,\delta}^+(x)$$

for any real x and

$$||A_{f,n,\delta}^+ - f||_1 \le \Big[V_{f_n^-}\frac{B_{n+1}(\beta_n)}{(n+1)!} - V_{f_n^+}\frac{B_{n+1}(\alpha_n)}{(n+1)!}\Big]\delta^{-n-1},$$

$$||f - A_{f,n,\delta}^-||_1 \le \Big[V_{f_n^+}\frac{B_{n+1}(\beta_n)}{(n+1)!} - V_{f_n^-}\frac{B_{n+1}(\alpha_n)}{(n+1)!}\Big]\delta^{-n-1}$$

with the functions $f_n^\pm$ given by (8).

After rescaling the exponential type of the approximation from $2\pi\delta$ to δ and an application of known estimates for the Bernoulli polynomials, Lemma 5 implies

$$E^\pm(f, \delta)_1 \le \pi V_{f^{(n)}}\delta^{-n-1}, \tag{9}$$

which is Theorem 1 for $p = 1$. (The necessary estimates for the Bernoulli polynomials can be found in inequalities (23.1.13), (23.1.14) and (23.1.15) of Abramowitz and Stegun [1].)

Proof of Lemma 5. We require some definitions. Let g be a measurable, bounded function on $\mathbb{R}$, and let h be an element of $L^1(-\infty, x)$ for every fixed $x \in \mathbb{R}$. We define for $x \in \mathbb{R}$

$$\mathcal{I}_+[g;h](x) := \int_0^\infty g(u)h(x-u)du, \tag{10}$$

$$\mathcal{I}_-[g;h](x) := \int_{-\infty}^0 g(u)h(u-x)du. \tag{11}$$

Assume now that $f^{(n)}$ is non-decreasing on $\mathbb{R}$. With the kernel

$$\mathfrak{K}_{n,\alpha,\delta}(x) := \frac{1}{n!}\delta^{-n}\mathscr{G}_{n,\alpha}(\delta x)$$

we define for even n

$$A^+_{f,n,\delta}(x) := \mathcal{I}_+[f^{(n)}; \mathfrak{K}'_{n,\alpha_n,\delta}](x) + \mathcal{I}_-[f^{(n)}; \mathfrak{K}'_{n,\beta_n,\delta}](x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j$$

and for odd n

$$A^+_{f,n,\delta}(x) := \mathcal{I}_+[f^{(n)}; \mathfrak{K}'_{n,\alpha_n,\delta}](x) - \mathcal{I}_-[f^{(n)}; \mathfrak{K}'_{n,\alpha_n,\delta}](x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j.$$

The lower approximations $A^-_{f,n,\delta}$ are defined by reversing the roles of α_n and β_n in the definition of $A^+_{f,n,\delta}$.

The function $f^{(n)}$ is bounded, since it has bounded variation. The function $\mathfrak{K}'_{n,\alpha,\delta}$ is not an element of $L^1(\mathbb{R})$ for $n \geq 1$, but by Lemma 3 it is an element of $L^1(-\infty, x)$ for every fixed $x \in \mathbb{R}$ and $\alpha \in \{\alpha_n, \beta_n\}$. These two facts show that the half-line convolutions $\mathcal{I}_\pm[f^{(n)}; \mathfrak{K}'_{n,\alpha,\delta}](x)$ converge absolutely for every real x.

Moreover, $A^{+(n)}_{f,n,\delta}$ consists of the sum of the half-line convolutions of $f^{(n)}$ with $\mathfrak{K}^{(n+1)}_{n,\alpha,\delta}$, and the latter function is integrable on $\mathbb{R}$ by Lemma 3. Hence the Fourier transform of $A^{+(n)}_{f,n,\delta}$ is continuous and bounded on $\mathbb{R}$. An investigation of its support and an application of the Paley-Wiener Theorem show that $A^{+(n)}_{f,n,\delta}$ and hence also $A^+_{f,n,\delta}$ are entire functions of exponential type $2\pi\delta$.

Define

$$\psi_{n,\alpha,\delta}(x) := \frac{1}{n!}(\delta^{-n}\mathscr{G}_{n,\alpha}(\delta x) - x^n_+).$$

Let n be even and $x > 0$. We have

$$\begin{aligned} f(x) - A^+_{f,n,\delta}(x) = -\int_0^\infty f^{(n)}(u) d\frac{(x-u)^n_+}{n!} &+ \int_0^\infty f^{(n)}(u) d\mathfrak{K}_{n,\alpha_n,\delta}(x-u) \\ &- \int_{-\infty}^0 f^{(n)}(u) d\mathfrak{K}_{n,\beta_n,\delta}(u-x), \end{aligned}$$

and combining the first two integrals into one integral, we obtain

$$\begin{aligned} f(x) - A^+_{f,n,\delta}(x) = \int_0^\infty f^{(n)}(u) d\Big[\mathfrak{K}_{n,\alpha_n,\delta}(x-u) &- \frac{(x-u)^n_+}{n!}\Big] \\ &- \int_{-\infty}^0 f^{(n)}(u) d\mathfrak{K}_{n,\beta_n,\delta}(u-x). \end{aligned}$$

Since $\mathfrak{K}_{n,\beta_n,\delta}(u-x) = \psi_{n,\beta_n,\delta}(u-x) = -\psi_{n,\alpha_n,\delta}(x-u)$ for $u - x < 0$ and even n (this follows from $x^n_+ = 0$ for $x < 0$ and from Lemma 2), we obtain with an integration by parts for even n and $x > 0$

$$\begin{aligned} f(x) - A^+_{f,n,\delta}(x) &= \int_{-\infty}^\infty f^{(n)}(u) d\psi_{n,\alpha_n,\delta}(x-u) \\ &= \int_{-\infty}^\infty \psi_{n,\alpha_n,\delta}(x-u) df^{(n)}(u) \leq 0, \end{aligned} \tag{12}$$

the last inequality follows from the fact that $f^{(n)}$ is assumed to be non-decreasing and $\psi_{n,\alpha_n,\delta} \leq 0$. Similar computations give (12) for $x < 0$. The $L^1(\mathbb{R})$ estimates follow after integration over x and an application of Lemma 1. The lower approximation is obtained by interchanging α_n and β_n. The computations for odd n and for f with non-increasing nth derivative are analogous.

If $f^{(n)}$ is not monotone, we consider its representation (8). We find the upper approximation for an nth antiderivative of f_n^+ and the lower approximation for an nth antiderivative of f_n^-. Up to a polynomial of degree $n-1$ (which has exponential type 0), their difference is an upper approximation of the function f.

The remaining statements of Theorem 1 are shown in the usual fashion. For $\alpha \in \{\alpha_n, \beta_n\}$, the function $\psi_{n,\alpha}$ defined in (6) is bounded on the real line, as can be seen from the representations in the proof of Lemma 3. We set

$$C_n := \max\left\{||\psi_{n,\alpha_n}||_\infty, ||\psi_{n,\beta_n}||_\infty\right\}.$$

For $0 \le a \le 1$ and $1 \le p < \infty$, the inequality $a^p \le a$ is valid and leads to the estimate $||h||_p^p \le ||h||_\infty^{p-1} ||h||_1$, provided $||h||_1$ and $||h||_\infty$ exist.

Assume first that $f^{(n)}$ is non-decreasing. In this case, (12) implies that $h := f - A^+_{f,n,\delta}$ satisfies

$$\begin{aligned} ||h||_\infty &\le ||\psi_{n,\alpha_n,\delta}||_\infty V_{f^{(n)}} = (n!)^{-1} ||\psi_{n,\alpha_n}||_\infty V_{f^{(n)}} \delta^{-n}, \\ ||h||_1 &\le ||\psi_{n,\alpha_n,\delta}||_1 \; V_{f^{(n)}} = (n!)^{-1} ||\psi_{n,\alpha_n}||_1 \; V_{f^{(n)}} \delta^{-n-1}. \end{aligned}$$

Inserting these estimates in the inequality for $||h||_p$ above gives Theorem 1 for f with non-decreasing and similarly for f with non-increasing nth derivative. The general statement follows by writing f as a sum of two functions with nth derivatives given by (8).

References

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University of British Columbia, Department of Mathematics, 1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada, email: flittman@math.ubc.ca, phone: 604 294 1170