ENTIRE FUNCTION MAJORANTS

BY

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THESIS

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To my parents

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Table of Contents

List of Figures

Chapter 1

Introduction and Notation

1.1 Introduction

Let $z = x + iy \in \mathbb{C}$ where x and y are real numbers. An entire function $F(z)$ is said to be of type δ if for every $\varepsilon > 0$

$$|F(z)| \leq A_\varepsilon \exp\big(|z|(\delta + \varepsilon)\big) \tag{1.1}$$

where $A_\varepsilon > 0$ is a constant depending on ε. (In the notation of [3] this a function of order 1 and type δ.) The set of all functions of type δ that are real on $\mathbb{R}$ will be denoted by $E(\delta)$.

The purpose of this thesis is to study a question of A. Beurling:

Question 1.1.1. *How well can a given function $f : \mathbb{R} \to \mathbb{R}$ be majorized or minorized on the real line by a real entire function of given exponential type δ?*

Let us discuss some examples which show how entire majorants and minorants have been used in the past.

In the 1930's Beurling studied Question 1.1.1 for the function $f(x) := \operatorname{sgn}(x)$, which is defined by $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$, and $\operatorname{sgn}(0) = 0$. He found that the entire function

$$B(z) := \frac{\sin^2 \pi z}{\pi^2}\Big(\sum_{n=0}^{\infty}(z-n)^{-2} - \sum_{n=-\infty}^{-1}(z-n)^{-2} + 2z^{-1}\Big) \tag{1.2}$$

satisfies the following extremal property:

(i) $B(x)$ majorizes the signum function $\operatorname{sgn}(x)$ for all $x \in \mathbb{R}$,

(ii) $B(z)$ is a function of type 2π,

(iii) $\int_{\mathbb{R}}(B - \mathrm{sgn}) = 1$,

(iv) any $F \in E(2\pi)$ with $F \geq \mathrm{sgn}$ on the real line and $F \neq B$ satisfies $\int_{\mathbb{R}}(F - \mathrm{sgn}) > 1$.

Beurling used this result to prove an inequality for real-valued, almost periodic trigonometric polynomials

$$f(x) := \sum_{n=1}^{N} a_n e(\lambda_n x) \tag{1.3}$$

satisfying $|\lambda_n| \geq \delta > 0$. (Here $e(y) = e^{2\pi i y}$.) He observed that the functions

$$\varphi_{1,\delta}(x) := \frac{1}{2}(\mathrm{sgn}(x) + B(-\delta x)) \geq 0, \tag{1.4}$$

$$\varphi_{2,\delta}(x) := -\frac{1}{2}(B(\delta x) - \mathrm{sgn}(x)) \leq 0, \tag{1.5}$$

are integrating kernels for the class of real-valued functions of the form (1.3). This means that the identities

$$f(x) = \varphi_{j,\delta} * f'(x) \text{ for } j = 1, 2 \tag{1.6}$$

hold for all functions $f(x)$ of the form (1.3), where $h*g(x) := \int_{\mathbb{R}} h(t)g(x-t)dt$ denotes the integral convolution. From the inequalities

$$f(x) \leq \sup_{y\in\mathbb{R}} f'(y) \int_{\mathbb{R}} \varphi_{1,\delta}(x)dx = \frac{1}{2\delta} \sup_{y\in\mathbb{R}} f'(y),$$

$$-f(x) \leq \sup_{y\in\mathbb{R}} f'(y) \int_{\mathbb{R}} (-\varphi_{2,\delta}(x))dx = \frac{1}{2\delta} \sup_{y\in\mathbb{R}} f'(y),$$

he concluded that any such function $f(x)$ satisfies

$$\sup_{x\in\mathbb{R}} |f(x)| \leq (2\delta)^{-1} \sup_{\xi\in\mathbb{R}} f'(\xi)$$

with the asymptotically sharp constant $(2\delta)^{-1}$. The remarkable feature of this inequality is that the absolute value is not needed on its right-hand side. Beurling presented this result in a seminar on Harmonic Analysis in 1942, but did not publish it before 1989 (cf. [2]); the first printed account of his work appeared in a survey article by J. D. Vaaler [25] in 1984.

The second example is a proof of the large sieve inequality found by A. Selberg. Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. We define a trigonometric polynomial by

$$P(x) := \sum_{\nu=m+1}^{m+n} a_\nu e(\nu x),$$

and consider real numbers $\xi_1, ..., \xi_r$ satisfying $||\xi_i - \xi_j|| \geq \delta > 0$ for all $i \neq j$. Here $||x||$ denotes the distance from the real number x to the nearest integer. The analytic formulation of the large sieve is an inequality of the form

$$\sum_{\rho=1}^{r} |P(\xi_\rho)|^2 \leq \Delta(n, \delta) \sum_{\nu=m+1}^{m+n} |a_\nu|^2. \tag{1.7}$$

Selberg discovered $B(z)$ independently, and he used it to give a proof of the large sieve inequality with the sharp bound $\Delta(n, \delta) = n - 1 + \delta^{-1}$ (cf. [22], chapter 20, [25], p. 185). His proof goes as follows. For $a, b \in \mathbb{R}$ define the function

$$C_{[a,b]}(x) := \frac{1}{2}\big(B(b-x) + B(x-a)\big).$$

Selberg noted that $F(x) := C_{[\delta(m+1), \delta(m+n)]}(\delta x)$ is an entire majorant of type $2\pi\delta$ of the characteristic function of the interval $[m+1, m+n]$ satisfying $\int_{\mathbb{R}} F = n - 1 + \delta^{-1}$. By a theorem of Fejér ([3], pp. 124–126) there is an entire function $f(z)$ such that $F = |f|^2$ on $\mathbb{R}$, and f has a Fourier transform $\widehat{f}$ supported in $[-\delta/2, \delta/2]$. We represent $P(x)$ in the form

$$P(x) = \sum_{\nu=m+1}^{m+n} f(\nu) \frac{a_\nu}{f(\nu)} e(\nu x) = \int_{-\delta/2}^{\delta/2} \widehat{f}(u) \sum_{\nu=m+1}^{m+n} \frac{a_\nu}{f(\nu)} e(\nu(x+u)) du.$$

Defining $P^*(x) := \sum_{\nu=m+1}^{m+n} (a_\nu / f(\nu)) e(\nu x)$, letting $x = \xi_\rho$ and applying Cauchy's inequality to the right-hand side of the previous identity we obtain

$$|P(\xi_\rho)|^2 \leq \int_{-\delta/2}^{\delta/2} |\widehat{f}(u)|^2 du \int_{-\delta/2}^{\delta/2} |P^*(\xi_\rho + u)|^2 du. \tag{1.8}$$

Note that the first integral on the right-hand side has value $\widehat{F}(0) = n - 1 + \delta^{-1}$.

Summing (1.8) over ρ, and using $||\xi_i - \xi_j|| \geq \delta$ for $i \neq j$, we obtain

$$\sum_{\rho=1}^{r} |P(\xi_\rho)|^2 \leq (n-1+\delta^{-1}) \int_0^1 |P^*(u)|^2 du.$$

By Parseval's equality for 1–periodic functions the last integral is equal to $\sum_{\nu=m+1}^{m+n} |a_\nu|^2 F(\nu)^{-1}$, and since $F(\nu) \geq 1$ for all $\nu \in [m+1, m+n]$ we obtain the large sieve inequality in the form (1.7) with $\Delta(n,\delta) = n - 1 + \delta^{-1}$.

As another application of entire majorants we can obtain sharp constants in certain inequalities related to the discrete Hilbert transform. Let $(a_\nu)_{\nu=1}^N$ be a finite sequence of complex numbers, and let $\{\lambda_\nu\}_{\nu=1}^N$ be a set of real numbers which are well-spaced in the sense that $|\lambda_\nu - \lambda_\mu| \geq \delta > 0$ for all $\nu \neq \mu$, and let $h(t)$ $(t \in \mathbb{R})$ be a hermitian function, i.e. $h(-t) = \overline{h(t)}$. We are interested in optimal bounds $L_\delta(h)$ and $U_\delta(h)$ such that

$$-L_\delta(h) \sum_{\nu=1}^{N} |a_\nu|^2 \leq \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{N} a_\nu \overline{a}_\mu h(\lambda_\nu - \lambda_\mu) \leq U_\delta(h) \sum_{\nu=1}^{N} |a_\nu|^2 \tag{1.9}$$

holds independently of $N \in \mathbb{N}$, and independently of the sequences $\{a_\nu\}_{\nu=1}^N$ and $\{\lambda_\nu\}_{\nu=1}^N$. If $h_1(t) := (it)^{-1}$, it can be shown that the best possible constants in (1.9) are given by $L(h_1) = U(h_1) = \pi\delta^{-1}$, and (1.9) is equivalent to

$$\Big| \sum_{\substack{m,n=1 \\ m\neq n}}^{N} \frac{a_m \overline{a}_n}{\lambda_m - \lambda_n} \Big| \leq \pi\delta^{-1} \sum_{n=1}^{N} |a_n|^2, \tag{1.10}$$

the so called Hilbert inequality. This estimate was first proved by H. Montgomery and R. C. Vaughan in [15]. Selberg showed that Beurling's majorant $B(z)$ can be used to give a very short proof of Montgomery and Vaughan's result (cf. [25] Theorem 16, [22] Chapter 20). In Section 4.3 we will give sharp constants in (1.9) for the class of functions $h_m(t) := (it)^{-m}$ where m is any positive integer.

This thesis is structured as follows. In Chapter 2 we present a way to construct extremal entire majorants and minorants of type $\delta > 0$ for $f_n(x) := \operatorname{sgn}(x)x^n$ $(n \in \mathbb{N}_0)$, i.e. for fixed $\delta > 0$ we will construct entire functions F_n and G_n with the properties

(i) F_n and G_n are entire functions of type δ,

(ii) $F_n - f_n \geq 0$ and $f_n - G_n \geq 0$ on the real line,

(iii) the integrals $\int (F_n - f_n)$ and $\int (f_n - G_n)$ are each minimal with respect to properties (i) and (ii).

The construction in Chapter 2 is based on a general method of obtaining candidates for extremal majorants given by S. W. Graham and Vaaler in [5]. Their approach is as follows. Assume for the moment that $f \in L^2(\mathbb{R})$, and that F is an entire function of type $\delta = 1$ satisfying $F \geq f$ on the real line. By the Paley-Wiener theorem (Theorem 2.2.1), the Fourier transform of F is supported on the interval $[-1, 1]$. Thus, letting $\psi := F - f$, the Poisson summation formula (Theorem 2.2.2) assumes the form

$$0 \leq \sum_{k\in\mathbb{Z}} \psi(x+k) = \sum_{\ell\in\mathbb{Z}} \widehat{\psi}(\ell)e(\ell x) = \widehat{\psi}(0) - \sum_{\ell\neq 0} \widehat{f}(\ell)e(\ell x), \tag{1.11}$$

for all $x \in [0, 1]$, which implies the necessary condition

$$\int_{\mathbb{R}} (F - f) = \widehat{\psi}(0) \geq \sup_{x\in[0,1]} \sum_{\ell\neq 0} \widehat{f}(\ell)e(\ell x). \tag{1.12}$$

Moreover, if we have equality in (1.12), then we obtain equality in (1.11) at $x = \alpha$, where α is the value at which the Fourier series on the right-hand side of (1.12) takes on its supremum. Since every summand in $\sum \psi(x + k)$ is non-negative, equality in (1.12) implies $\psi(\alpha + k) = 0$ for all k, i.e.

$$F(\alpha + k) = f(\alpha + k) \text{ for all } k \in \mathbb{Z}. \tag{1.13}$$

The interpolating property (1.13) suggests how to construct entire majorants of type $\delta = 1$ of $f_n(x) := \operatorname{sgn}(x)x^n$, even though the functions f_n are not integrable. The extension to arbitrary fixed type δ will follow from homogeneity properties of f_n.

In Chapter 3 we present a different method of constructing entire approximations based on properties of the two-sided Laplace transform. This was first introduced by J. J. Holt and Vaaler in [8] in order to develop extremal entire majorants of the characteristic function of the unit ball in $\mathbb{R}^n$.

We extend their idea to give entire extremal majorants and minorants of

$\operatorname{sgn}(x)P(x)$ for certain classes of polynomials $P(x)$. Define for $g : \mathbb{R} \to \mathbb{R}$ the bilateral Laplace transform of g by

$$\mathcal{L}[g](z) := \int_{-\infty}^{\infty} g(t)e^{-zt}dt \tag{1.14}$$

for all $z \in \mathbb{C}$ for which the integral converges absolutely. A useful feature of the Laplace transform is that it transforms the action of a differential operator $P(D)$ with constant coefficients into multiplication by the polynomial $P(z)$.

In Chapter 3 we use this property in the opposite direction. Consider F and g such that

$$F^{-1}(z) = \mathcal{L}[g](z) \tag{1.15}$$

holds in some vertical strip of the complex plane. For technical reasons we use the pair $z^{-1}F(z)$ and $g'(t)$ rather than $F(z)$ and $g(t)$, i.e. we define the entire approximation to $\operatorname{sgn}(x)P(x)$ by

$$G_P(z) := -z^{-1}F(z)\mathcal{L}[\operatorname{sgn} \cdot P(D)g'](z), \tag{1.16}$$

where $D := d/dt$ denotes the differential operator. We will see that the presence of any one of the two conditions $F(0) = 0$ or $P(D)g(0) = 0$ implies that $G_P(z)$ is entire and satisfies the growth estimate

$$|G_P(x) - \operatorname{sgn}(x)P(x)| \ll (1 + |x|)^{-2}F(x) \text{ as } |x| \to \infty$$

In particular, if F is bounded, then the difference $G_P(x) - \operatorname{sgn}(x)P(x)$ is absolutely integrable.

In order to obtain entire majorants and minorants for $\operatorname{sgn}(x)P(z)$ we take the functions $F(z)$ to be elements of the class of Pólya-Laguerre functions. These entire functions have the property that they are real-valued on the real line, have only real zeros, and are uniform limits of polynomials having only real roots.

We restrict ourselves to these functions because the function g associated to a Pólya-Laguerre function F via (1.15) satisfies the variation diminishing property, i.e. for any bounded, continuous, real-valued function φ, the integral convolution $g * \varphi(x) := \int_{\mathbb{R}} g(t)\varphi(x-t)dt$ has no more changes of sign on the real line than φ has. We show in Section 3.3 that this property implies that for certain classes of polynomials the term $x^{-1}\mathcal{L}[\mathrm{sgn}\cdot P(D)g'](x)$ has no changes of sign on the real line. By taking F to be non-negative and bounded on the real line, we obtain entire majorants and minorants of $\mathrm{sgn}(x)P(z)$.

In Chapter 4 we give four applications. In Section 4.1 we give an extension of Beurling's result mentioned above to arbitrary derivatives. We show that functions of the form (1.3) satisfy

$$|f(x)| \leq \max\{B_{m,\delta}(\alpha_{m-1}), -B_{m,\delta}(\beta_{m-1})\} \sup_{x\in\mathbb{R}} f^{(m)}(x), \tag{1.17}$$

with $B_{m,\delta}(\alpha_{m-1})$ defined in (4.10) and α_m, β_m defined in (2.33).

Define functions $f_m(t) := \pm m!(2\pi i t)^m$ restricted to $\mathbb{R}\backslash[-1,1]$. In Section 4.2 we construct positive definite extensions of $f_m(t)$ to $\mathbb{R}$ (i.e. the Fourier transforms of these extensions are non-negative on $\mathbb{R}$) such that $f_m(0)$ is minimal among all such extensions.

In Section 4.3 we use the results of Section 4.2 to prove the generalizations of Hilbert's inequality mentioned above.

In Section 4.4 we give entire majorants for functions of the form

$$R_\alpha(x) := \alpha^{-1}(|x+\alpha| - |x|) \text{ for all } x \in \mathbb{R}.$$

Entire majorants for these functions were originally obtained by J. J. Holt [7].

1.2 Notation

Define

$$\begin{aligned}
\mathbb{Z} &: \text{ integers} \\
\mathbb{N} &: \text{ positive integers} \\
\mathbb{N}_0 &: \text{ non-negative integers} \\
\mathbb{R} &: \text{ real numbers} \\
\mathbb{C} &: \text{ complex numbers} \\
L^2(\mathbb{R}) &: \text{ functions which are square integrable over } \mathbb{R} \\
f * g(x) &:= \int_{-\infty}^{\infty} f(t)g(x-t)dt \\
C^\infty(\mathbb{R}) &: \text{ functions which are infinitely differentiable on } \mathbb{R} \\
Df(x) &:= \frac{d}{dx} f(x) \\
Z_S[f] &: \text{ number of zeros of } f \text{ in } S\text{, counted with multiplicity} \\
E(2\pi\delta) &: \text{ entire functions of type } 2\pi\delta \text{ (cf. (1.1))} \\
\mathcal{E} &: \text{ Pólya-Laguerre entire functions, cf. Definition 3.2.2} \\
T_a f(x) &:= |a| f(ax) \text{ for } a \in \mathbb{R} \\
[x] &: \text{ largest integer in } x \in \mathbb{R} \\
\{x\} &:= x - [x] \\
B_n(t) &: \ n\text{th Bernoulli polynomial, cf. Section 2.3} \\
\mathcal{B}_n(t) &: \ n\text{th periodic Bernoulli function } \mathcal{B}_n(t) = B_n(t - [t]) \\
z_{2n} &: \text{ the zero of } B_{2n}(t) \text{ in the interval } (0, 1/2) \\
\alpha_n &: \text{ maximum of } \mathcal{B}_n(t) \\
\beta_n &: \text{ minimum of } \mathcal{B}_n(t) \\
H_n(z; \alpha) &: \text{ cf. Definition 2.5.1} \\
\psi_{n,\alpha}(x) &:= H_n(x; \alpha) - \operatorname{sgn}(x) x^n, \text{ cf. (2.37)} \\
\varphi_{\delta,m,\alpha}(x) &:= \frac{1}{2(m-1)!} (\operatorname{sgn}(x) x^{m-1} - \delta^{-m+1} H_{m-1}(\delta x; \alpha)) \\
\varphi^+_{\delta,m} &:= \varphi_{\delta,m,\alpha_{m-1}} \\
\varphi^-_{\delta,m} &:= \varphi_{\delta,m,\beta_{m-1}}
\end{aligned}$$

$$g_{(n)}(t) := g(t) - \sum_{j=0}^{n} \frac{g^{(j)}(0)}{j!} t^j$$

$$G_{F,P}(z) := -\frac{F(z)}{z}\mathcal{L}[\mathrm{sgn}\cdot P(D)g'](z), \text{ cf. Definition 3.5.1}$$

$$G_P(z) : \text{ stands for } G_{F,P}(z) \text{ if } F(z) \text{ is unambiguous}$$

$$G_n(z) : \text{ stands for } G_{F,x^n}(z) \text{ if } F(z) \text{ is unambiguous}$$

$$F_\alpha(z) := \frac{\sin^2 \pi(z-\alpha)}{\pi^2}$$

$$g_\alpha(t) := te^{\alpha t}(e^t - 1)^{-1}$$

$$G_{\alpha,n}(z) : \text{ stands for } G_{F_\alpha,x^n}(z)$$

$$S_F := \{b \in \mathbb{R} \,:\, F(b) = 0\} \cup \{\pm\infty\}$$

$$\mathrm{sgn}(x) : \text{ the normalized signum - function}$$

$$\mathrm{sgn}_+(x) := \lim_{t\to x+} \mathrm{sgn}(t)$$

$$\mathrm{sgn}(x,t) := \mathrm{sgn}(x) + \mathrm{sgn}(t)$$

$$\mathfrak{g}(t) : \text{ equals } e^t \text{ on } t < 0 \text{ and } 0 \text{ on } t > 0$$

$$\mathcal{V}[a_1, ..., a_n] : \text{ number of sign changes in the sequence } a_1, ..., a_n$$

$$\mathcal{V}[f] := \sup\{\mathcal{V}[f(x_1), ..., f(x_n) \,|\, n \in \mathbb{N}, -\infty < x_1 < ... < x_n < \infty\}$$

The symbol $F^{-1}(x)$ always denotes the reciprocal, and never the functional inverse of $F(x)$.

We define the Fourier transform of a function $f \in L^2(\mathbb{R})$ by

$$\widehat{f}(t) := \lim_{N\to\infty} \int_{-N}^{N} f(x)e(-tx)dx, \tag{1.18}$$

here we use the notation $e(y) = \exp(2\pi i y)$.

If $f \in L^1([0,1])$ we define for $k \in \mathbb{Z}$ the kth Fourier coefficient b_k of f by

$$b_k := \int_0^1 f(x)e(-xk)dx. \tag{1.19}$$

We use the following symbols to indicate growth estimates. By writing

$$f(x) = \mathcal{O}\big(g(x)\big) \text{ on } M \subseteq \mathbb{C},$$

or

$$f(x) \ll g(x) \text{ on } M \subseteq \mathbb{C}.$$

for a function $g \geq 0$, we mean that there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in M$. If $g(x) \neq 0$ on M we write

$$f(x) = o(g(x)) \text{ as } x \to a \text{ in } M \subseteq \mathbb{C}$$

to indicate that $f(x)/g(x) \to 0$ as x converges to a through elements in M.

Chapter 2

Extremal Functions via Euler – Maclaurin Summation

2.1 Introduction

In this chapter we use the classical method of Beurling and Selberg to construct entire majorants for the class of functions

$$f_n(x) := \operatorname{sgn}(x)x^n,$$

where $n \in \mathbb{N}_0$. Variants of this method have been used in [2], [5], [7], [22], and [25]. The idea is the following:

The Paley-Wiener theorem (Theorem 2.2.1) asserts that functions in $E(2\pi\delta) \cap L^2(\mathbb{R})$ have a Fourier transform with support in $[-\delta, \delta]$.

The (distributional) Fourier transform of $f_n(x) := \operatorname{sgn}(x)x^n$ is given by

$$\widehat{f}_n(t) = \frac{2 \cdot n!}{(2\pi i t)^{n+1}}.$$

Assume that $M_n(z)$ is a function of type 2π which is real-valued on the real line and has the property that the difference

$$\psi_n(x) := M_n(x) - f_n(x) \quad (x \in \mathbb{R})$$

is integrable. In view of the Paley-Wiener theorem we can expect that the Fourier transform of ψ_n depends on M_n only in the open unit interval $[-1, 1]$, and in particular $\widehat{\psi}_n(\ell) = -2 \cdot n!\,(2\pi i\ell)^{-n-1}$ for all integers $\ell \neq 0$. Using the

Poisson summation formula (Theorem 2.2.2) we obtain

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\psi_n(x+k) &= \widehat{\psi}_n(0)-\frac{2\cdot n!}{(2\pi i)^{n+1}}\sum_{\ell\neq 0}\frac{1}{\ell^{n+1}}e(\ell x)\\ &= \int_{\mathbb{R}}(M_n-f_n)+\frac{2}{n+1}\mathcal{B}_{n+1}(x)\end{aligned}\tag{2.1}$$

with the periodic Bernoulli function $\mathcal{B}_{n+1}(x) := B_{n+1}(x-[x])$, where $[x]$ denotes the greatest integer not exceeding x, and B_n is the nth Bernoulli polynomial (cf. Lemma 2.3.1).

If $M_n \geq f_n$ on the real line, the right-hand side in (2.1) must be non-negative. In particular,

$$\int_{\mathbb{R}}(M_n-f_n) \geq -\frac{2}{n+1}\inf_{0\leq x\leq 1}\mathcal{B}_{n+1}(x). \tag{2.2}$$

Assuming that we can find a real entire function M_n of type 2π which majorizes f_n on the real line such that we have equality in (2.2), the right-hand side of (2.1) must equal zero at the point $x=\alpha$, where $B_{n+1}(x)$ assumes its minimum on $[0,1]$. Since $\psi_n(k+\alpha)\geq 0$ for all integers k, this can be possible only if

$$M_n(\alpha+k)=f_n(\alpha+k) \text{ for all } k\in\mathbb{Z}. \tag{2.3}$$

(The existence of such a function M_n is *a priori* not guaranteed.)

In order to construct entire majorants and minorants for the functions $f_n(x) := \operatorname{sgn}(x)x^n$ we will start by defining entire functions of type 2π which satisfy the interpolation property (2.3). We will then show that these functions will be majorants and minorants of $\operatorname{sgn}(x)x^n$ provided that α is the value where $B_{n+1}(x)$ assumes an extremum on $[0,1]$. Moreover, we will prove that the functions M_n chosen in this way satisfy

$$\int_{\mathbb{R}}(M_n-f_n) = -\frac{2}{n+1}B_{n+1}(\alpha). \tag{2.4}$$

The heuristic consideration in (2.1) will be made rigorous in the proof of Lemma 2.5.6, where we will show that the value in (2.4) is optimal.

The remaining parts of this chapter are organized as follows. Sections 2.2, 2.3, and 2.4 contain auxiliary results about entire functions of finite exponential type, Bernoulli functions, and interpolation theorems for functions in $E(2\pi k) \cap L^2(\mathbb{R})$, respectively. These results are needed later. In Section 2.5 the majorants and minorants of $\mathrm{sgn}(x)x^n$ will be given.

2.2 Entire Functions of Finite Type

This section contains three classical results dealing with functions of finite type and with Fourier transforms of functions in $L^1(\mathbb{R})$, namely the Paley-Wiener Theorem, the Poisson summation formula, and a result of Plancherel and Pólya connecting $\sum |F(n)|^2$ and $\int |F|^2$ for integrable entire functions $F(z)$ of finite type.

The Paley-Wiener theorem describes functions of exponential type in terms of the support their Fourier transforms. Formulation and proof are taken from [18], Theorem 19.3.

Lemma 2.2.1 (Paley-Wiener). *Suppose A and C are positive constants and f is an entire function such that*

$$|f(z)| \leq Ce^{2\pi A|z|} \tag{2.5}$$

for all $z \in \mathbb{C}$, and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty. \tag{2.6}$$

Then there exists a function $F \in L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} F(t)e^{2\pi iAt}dt$$

for all $z \in \mathbb{C}$.

Sketch of proof. For $\theta \in \mathbb{R}$ define the ray Γ_θ by

$$\Gamma_\theta(s) := se^{i\theta} \text{ where } 0 \leq s < \infty.$$

Also define

$$\Phi_\theta(w) := \int_{\Gamma_\theta} f(z)e^{-wz}dz = e^{i\theta}\int_0^{\infty} f(se^{i\theta})e^{-wse^{i\theta}}ds. \tag{2.7}$$

By (2.5) we have

$$|f(se^{i\theta})e^{-wse^{i\theta}}| \ll e^{2\pi A|s| + \Re(-wse^{-i\theta})}.$$

Therefore the integral defining Φ_θ is absolutely convergent for all $w \in \Pi_\theta := \{z : \Re(ze^{i\theta}) > 2\pi A\}$ and $\Phi_\theta(w)$ is holomorphic in this half-plane. Moreover, by (2.6) the function $\Phi_0(w)$ is holomorphic in $\Re w > 0$, and the function $\Phi_\pi(w) = \Phi_{-\pi}(w)$ is holomorphic in $\Re w < 0$.

Let w be of the form $w = |w|e^{-i\pi/4}$ such that $w \in \Gamma_0 \cap \Gamma_{\pi/2}$. (These points w satisfy $|w| > \sqrt{2}A$).) Form a contour Γ_K consisting of the union of $\Gamma_0(s)$ and $\Gamma_{\pi/2}(s)$ with $0 \le s < K$ and the circular arc $C_K(\varphi) := Ke^{i\varphi}$ where $0 \le \varphi \le \pi/2$, and integrate $f(z)e^{-wz}$ over Γ_K.

It can be shown that $\int_{C_K} f(z)e^{-wz}dz \to 0$ as $K \to \infty$, and so Φ_0 and $\Phi_{\pi/2}$ are analytic continuations of each other. With a similar argument $\Phi_{\pi/2}$ and Φ_π can be shown to be analytic continuations of each other.

We have for $\varepsilon > 0$ and $t < -A$

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)e^{-\varepsilon|x|}e^{-2\pi itx}dx &= \Phi_0(2\pi it - \varepsilon) - \Phi_\pi(2\pi it + \varepsilon)\\ &= \Phi_{\pi/2}(2\pi it - \varepsilon) - \Phi_{\pi/2}(2\pi it + \varepsilon)\\ &\to 0 \text{ as } \varepsilon \to 0+\,.\end{aligned}$$

Therefore the Fourier transform of f equals zero for $t < -A$. After replacing $\Phi_{\pi/2}$ by $\Phi_{-\pi/2}$ and Φ_π by $\Phi_{-\pi}$, the same argument shows that the Fourier transform of f equals zero for $t > A$. $\square$

Recall that the Fourier coefficients of an $L^1([0,1])$–function f are defined by

$$b_k := \int_0^1 f(x)e^{-2\pi ixk}dx \text{ for } k \in \mathbb{Z}.$$

To every 1–periodic, integrable function we assign its formal Fourier series $\sum_k b_k e(xk)$. (Recall that $e(y) := e^{2\pi iy}$.)

We remark that $f(x) = \sum_k b_k e(xk)$ for all x in the interval $[0,1]$, provided $\sum_k |b_k|$ converges uniformly (cf. [23], VII.1.8). If $f \in L^1 \cap L^2([0,1])$, then

$$\lim_{N\to\infty} \Big|\Big| \sum_{k=-N}^{N} b_k e(xk) - f(x)\Big|\Big|_2 = 0,$$

where $||.||_2$ denotes the norm of $L^2([0,1])$. In particular, the symmetric partial sums converge to $f(x)$ almost everywhere (cf. [23], proof of VII.1.7).

Let $f \in L^1(\mathbb{R})$. The Poisson summation formula connects two ways of obtaining a 1–periodic function from f. The first construction is elementary. Write

$$\phi(x) := \sum_{m\in\mathbb{Z}} f(x+m). \tag{2.8}$$

Since this formal sum is taken over all integers, the function ϕ is 1–periodic, if the series converges. For the second approach let $\widehat{f}(t)$ be the Fourier transform of f given by

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e(-xt)dx,$$

and define a formal Fourier series by

$$\sum_{m\in\mathbb{Z}} \widehat{f}(m)e(mx). \tag{2.9}$$

The Poisson summation formula in the form in which we will need it states that the two approaches to obtain a periodic analogue of f lead essentially to the same function in $L^1(\mathbb{R}/\mathbb{Z})$. The statement of the Poisson summation in the following theorem is taken from [23] Theorems 2.4 and 2.11.

Let $K(x) = 1 - |x|$ for $|x| \leq 1$ and $K(x) = 0$ for $|x| > 1$. Set $K_\varepsilon(x) = \varepsilon^{-1}K(\varepsilon^{-1}x)$.

Lemma 2.2.2. *For $f \in L^1(\mathbb{R})$ the series (2.8) converges in the norm of $L^1(\mathbb{R}/\mathbb{Z})$. The resulting function has the Fourier expansion (2.9). Moreover, letting*

$$\varphi_\varepsilon(x) := \sum_m \widehat{K}(\varepsilon m)\widehat{f}(m)e(mx), \tag{2.10}$$

the symmetric partial sums of the series ϕ defined by (2.8) converge to $\varphi := \lim_{\varepsilon\to 0}\varphi_\varepsilon$ in the norm of $L^1([0,1])$. In addition, this convergence is pointwise if f is normalized, i.e. $f(x) = (f(x+) + f(x-))/2$ for all real numbers x.

Sketch of proof. Since $f \in L^1(\mathbb{R})$, the integral $\int_{[0,1]} |\sum_m f(x+m)|dx \leq \int |f| < \infty$ exists (the sum is taken over all integers m), and so $\phi(x) =$

$\sum_m f(x+m)$ converges absolutely in the norm of $L^1([0,1])$. A similar argument can be used to show that the Fourier series coefficients of ϕ exist and equal the values of the Fourier transform of f at the integers.

The convergence properties of Lemma 2.2.2 are established by convolving ϕ with the 1-periodic function $\sum_m K_\varepsilon(x+m)$. This convolution has the Fourier expansion (2.10), and it can be shown that the statements of the lemma are obtained by letting $\varepsilon \to 0$. $\square$

Finally, we will need the following result of Plancherel and Pólya [17]:

Lemma 2.2.3. *If $F \in E(2\pi) \cap L^2(\mathbb{R})$, then*

$$\sum_{m=-\infty}^{\infty} |F(m)|^2 \leq \frac{2}{\pi^2}(e^{2\pi}-1)\int_{-\infty}^{\infty} |F(x)|^2 dx.$$

Sketch of proof. Let $\zeta \in \mathbb{C}$ and $\delta > 0$. We start with Cauchy's integral formula in the form $F(\zeta) = (2\pi)^{-1}\int_0^{2\pi} F(\zeta + re^{i\varphi})d\varphi$, valid in any circle around ζ with radius r, and apply the Cauchy-Schwarz inequality to obtain the inequality $|F(\zeta)|^2 \leq (2\pi)^{-1}\int_0^{2\pi} |F(\zeta+re^{i\varphi})|^2 d\varphi$. Integrating r from 0 to δ on both sides gives

$$\begin{aligned} |F(\zeta)|^2 \int_0^\delta r dr &\leq \frac{1}{2\pi}\int_0^{2\pi}\int_0^\delta |F(\zeta+re^{i\varphi})|^2 r dr d\varphi \\ &= \frac{1}{2\pi}\int_{x^2+y^2\leq\delta^2} |F(\zeta+x+iy)|^2 d(x,y). \end{aligned}$$

After replacing the circle of integration with the larger rectangle $[-\delta,\delta]^2$ and dividing by $\int_0^\delta r dr = \delta^2/2$, we find that

$$|F(\zeta)|^2 \leq \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |F(\zeta+x+iy)|^2 dx dy.$$

We apply this inequality with $\zeta = m$ and $\delta = 1/2$ and sum over all integers m. We obtain

$$\begin{aligned} \sum_{m=-\infty}^{\infty} |F(m)|^2 &\leq \frac{4}{\pi}\sum_{m=-\infty}^{\infty}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |F(m+x+iy)|^2 dx dy \\ &= \frac{4}{\pi}\int_{-1/2}^{1/2}\int_{-\infty}^{\infty} |F(x+iy)|^2 dx dy. \end{aligned} \tag{2.11}$$

The changes of integration in (2.11) are justified, since all integrands are non-negative.

Recall that $F(z) = \int_{-1}^{1} \widehat{F}(t)e(zt)dt$ by Theorem 2.2.1. With $z = x + iy$ ($x, y \in \mathbb{R}$) this can be written as $F(x+iy) = \int_{-1}^{1} \widehat{F}(t)e^{-2\pi yt}e^{2\pi ixt}dt$, and Plancherel's formula gives the estimate

$$\begin{aligned}\int_{-\infty}^{\infty} |F(x+iy)|^2 dx = \int_{-1}^{1} |\widehat{F}(t)|^2 e^{-4\pi yt} dt &\leq e^{4\pi|y|} \int_{-1}^{1} |\widehat{F}(t)|^2 dt \\ &= e^{4\pi|y|} \int_{-\infty}^{\infty} |F(x)|^2 dx.\end{aligned}$$

Inserting this inequality in (2.11) gives

$$\begin{aligned}\sum_{m=-\infty}^{\infty} |F(m)|^2 &\leq \frac{4}{\pi} \int_{-1/2}^{1/2} e^{4\pi|y|} dy \int_{-\infty}^{\infty} |F(x)|^2 dx \\ &= \frac{2}{\pi^2}(e^{2\pi} - 1) \int_{-\infty}^{\infty} |F(x)|^2 dx,\end{aligned}$$

which completes the proof. □

2.3 Bernoulli Functions and Euler-Maclaurin Summation

In this section we give a brief review of certain properties of Bernoulli polynomials that we will need in later proofs. These results are taken from [1], [13], and [16].

The Bernoulli polynomials $B_n(x)$ can be defined by the power series expansion

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \tag{2.12}$$

valid for $|t| < 2\pi$, and the Bernoulli periodic functions $\mathcal{B}_n(t)$ by

$$\mathcal{B}_n(t) = B_n(t - [t]). \tag{2.13}$$

Expanding $t^2 e^{xt}(e^t - 1)^{-1}$ in the region $|t| < 2\pi$ in a Maclaurin series shows

$$B_n'(x) = nB_{n-1}(x). \tag{2.14}$$

Integrating (2.12) from $x = 0$ to $x = 1$ proves

$$\int_0^1 B_n(x)dx = 0 \text{ for } n \geq 1. \tag{2.15}$$

Evaluation of (2.12) at $x = 0$ and $x = 1$ and a comparison of coefficients shows

$$B_n(0) = B_n(1) \text{ for } n \geq 2. \tag{2.16}$$

.

Equations (2.13), (2.14), (2.15), and (2.16) imply that for $0 \leq \alpha \leq 1$, $n \in \mathbb{N}$, and $x \in \mathbb{R}$ the relation

$$\int_0^x \mathcal{B}_n(t+\alpha)dt = \frac{1}{n+1}\big(\mathcal{B}_{n+1}(x+\alpha) - B_{n+1}(\alpha)\big) \tag{2.17}$$

holds.

Evaluating (2.12) at $x = \alpha$ and $x = 1 - \alpha$ shows

$$B_n(\alpha) = (-1)^n B_n(1-\alpha). \tag{2.18}$$

The first Bernoulli polynomials are

$$\begin{array}{rclrcl} B_0(x) & = & 1, & B_3(x) & = & x^3 - 3x^2/2 + x/2, \\ B_1(x) & = & x - 1/2, & B_4(x) & = & x^4 - 2x^3 + x^2 - 1/30, \\ B_2(x) & = & x^2 - x + 1/6, & B_5(x) & = & x^5 - 5x^4/2 + 5x^3/3 - x/6. \end{array}$$

Lemma 2.3.1. *For $n \geq 1$ the Bernoulli periodic functions have the Fourier series expansion*

$$\mathcal{B}_n(t) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^n} e(kt), \tag{2.19}$$

which is valid for $t \in \mathbb{R}\backslash\mathbb{Z}$ with symmetric summation, and it is uncondi-

tionally convergent for $t \in \mathbb{R}$ if $n \geq 2$.

Sketch of proof. We have $\mathcal{B}_1(x) = x - 1/2$ for $0 < x < 1$. Recall that the Fourier coefficients of an $L^1([0,1])$–function f are defined by

$$b_k := \int_0^1 f(x) e^{-2\pi i x k} dx \text{ for } k \in \mathbb{Z}.$$

We can verify directly that $\mathcal{B}_1(x) = x - 1/2$ has the Fourier coefficients $b_0 = 0$ and $b_k = -(2\pi i k)^{-1}$ if $k \neq 0$. The series $\sum_k b_k e(xk)$ does not converge absolutely, but it can be shown that

$$\mathcal{B}_1(x) = -\lim_{N\to\infty} \sum_{m=-N}^{N} (2\pi i k)^{-1} e(kx) \text{ for } 0 < x < 1. \tag{2.20}$$

Moreover, it can be shown that the symmetric limit in (2.20) is uniformly bounded in N and x, so the termwise integration of (2.20) is justified. An induction using (2.14) and (2.15) completes the proof. □

We will need the extrema of the Bernoulli polynomials in the interval $[0,1]$. The locations of these extrema are given in the following lemma. These facts come from [16], Chapter 2.

Lemma 2.3.2. *Let $0 \leq x \leq 1$ and $n \geq 1$.*

(i) $B_{4n}(x)$ assumes its maximum value at $x = 1/2$ and its minimum value at $x = 0$ and $x = 1$.

(ii) $B_{4n+1}(x)$ assumes its minimum value at a unique $\alpha \in (0, 1/2)$ and its maximum value at $1 - \alpha \in (1/2, 1)$.

(iii) $B_{4n-2}(x)$ assumes its maximum value at $x = 0$, $x = 1$ and its minimum value at $x = 1/2$.

(iv) $B_{4n-1}(x)$ assumes its maximum value at a unique $\alpha \in (0, 1/2)$ and its minimum value at $1 - \alpha \in (1/2, 1)$.

Below are plots of $B_1(x), ..., B_6(x)$ on the interval $[0,1]$.

Remark 2.3.3. D. H. Lehmer showed in [12] that the zeros z_{2n} of the even Bernoulli polynomial $B_{2n}(x)$ in $(0, 1/2)$ (or, what amounts to the same thing,

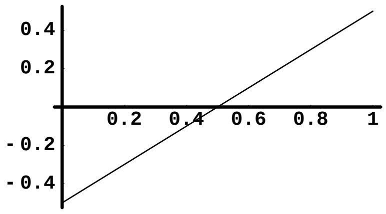

Figure 2.1: $B_1(t)$ on $[0,1]$

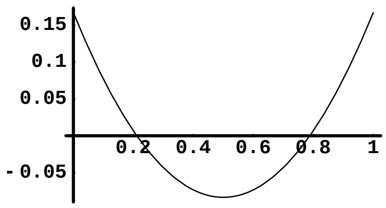

Figure 2.2: $B_2(t)$ on $[0,1]$

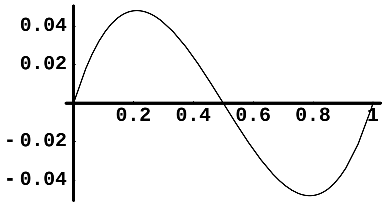

Figure 2.3: $B_3(t)$ on $[0,1]$

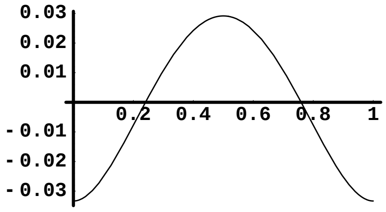

Figure 2.4: $B_4(t)$ on $[0,1]$

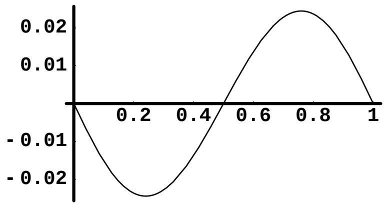

Figure 2.5: $B_5(t)$ on $[0,1]$

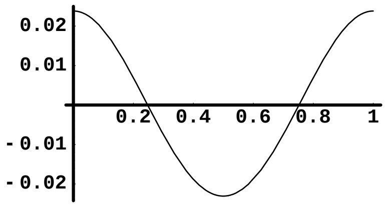

Figure 2.6: $B_6(t)$ on $[0,1]$

the extrema of the odd Bernoulli polynomial $B_{2n+1}(x)$ in $(0, 1/2)$) satisfy

$$\frac{1}{4} - \frac{1}{\pi 2^{2n+1}} < z_{2n} < \frac{1}{4}.$$

Decimal approximations for the first four z_{2n} are $z_2 = 0.2113$, $z_4 = 0.2403$, $z_6 = 0.2475$, $z_8 = 0.2494$.

The following lemma is an application of the Euler-Maclaurin summation formula.

Lemma 2.3.4. *For $0 \le \alpha \le 1$, $x > 0$, and any $\mu \in \mathbb{N}$*

$$\sum_{k=1}^{\infty} \frac{1}{(x+k-\alpha)^2} = \sum_{\nu=1}^{\mu} \frac{B_{\nu-1}(\alpha)}{x^{\nu}} + (\mu+1) \int_0^{\infty} \frac{B_{\mu}(\alpha) - \mathcal{B}_{\mu}(t+\alpha)}{(x+t)^{\mu+2}} dt. \tag{2.21}$$

Proof. We will use an induction on μ. For $0 \le \alpha < 1$ we obtain with integration by parts

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{(x+k-\alpha)^2} &= \int_{0+}^{\infty} \frac{1}{(x+t)^2} d[t+\alpha] \\ &= \int_0^{\infty} \frac{1}{(x+t)^2} dt + \int_{0+}^{\infty} \frac{d[t+\alpha] - dt}{(x+t)^2} \\ &= \frac{B_0(\alpha)}{x} + 2 \int_0^{\infty} \frac{B_1(\alpha) - \mathcal{B}_1(t+\alpha)}{(x+t)^3} dt, \end{aligned} \tag{2.22}$$

where we used $B_1(\alpha) - \mathcal{B}_1(t+\alpha) = [t+\alpha] - t$ and $\lim_{t\to 0+}(B_1(\alpha) - \mathcal{B}_1(t+\alpha)) = 0$. To show (2.21) for $\mu = 1$ and $\alpha = 1$ we apply (2.22) with $\alpha = 0$:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{(x+k-1)^2} &= \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} + \frac{1}{x^2} = \frac{B_0(0)}{x} + 2 \int_0^{\infty} \frac{B_1(0) - \mathcal{B}_1(t) + 1}{(x+t)^3} dt \\ &= \frac{B_0(1)}{x} + 2 \int_0^{\infty} \frac{B_1(1) - \mathcal{B}_1(t+1)}{(x+t)^3} dt, \end{aligned}$$

since $1 + B_1(0) = 1/2 = B_1(1)$, $B_0(0) = 1 = B_0(1)$, and $\mathcal{B}_1(t)$ is 1-periodic. This establishes (2.21) for $\mu = 1$.

Now assume that the lemma is true for $\mu \in \mathbb{N}$ and $0 \le \alpha \le 1$. We obtain with (2.17)

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{(x+k-\alpha)^2} &= \sum_{\nu=1}^{\mu}\frac{B_{\nu-1}(\alpha)}{x^\nu}+(\mu+1)\int_0^\infty\frac{B_\mu(\alpha)-\mathcal{B}_\mu(t+\alpha)}{(x+t)^{\mu+2}}dt\\
&= \sum_{\nu=1}^{\mu}\frac{B_{\nu-1}(\alpha)}{x^\nu}+\frac{B_\mu(\alpha)}{x^{\mu+1}}+\int_0^\infty\frac{-(\mu+1)\mathcal{B}_\mu(t+\alpha)}{(x+t)^{\mu+2}}dt\\
&= \sum_{\nu=1}^{\mu+1}\frac{B_{\nu-1}(\alpha)}{x^\nu}+(\mu+2)\int_0^\infty\frac{B_{\mu+1}(\alpha)-\mathcal{B}_{\mu+1}(t+\alpha)}{(x+t)^{\mu+3}}dt,
\end{aligned}$$

which is the claim of the lemma for $\mu+1$. □

2.4 Interpolation Theorems

In Lemma 2.5.6 we will prove that the extremal majorants and minorants of $\operatorname{sgn}(x)x^n$ are unique. This fact will be shown using an interpolation theorem of Vaaler ([25], Theorem 9), which states that every $F\in E(2\pi)\cap L^2(\mathbb{R})$ has the unique representation

$$F(z)=\frac{\sin^2(\pi z)}{\pi^2}\Big(\sum_{k=-\infty}^{\infty}\frac{F(k)}{(z-k)^2}+\sum_{k=-\infty}^{\infty}\frac{F'(k)}{z-k}\Big).$$

In this section we will prove a generalization of this theorem to functions $F\in E(2\pi k)$ where $k\in\mathbb{N}$. We will start by constructing a family of kernels which will enable us to give an interpolation formula for entire functions of finite exponential type.

Definition 2.4.1. Define for $\nu\in\mathbb{N}$ and $0\le j<\nu$

$$K_{\nu,j}(z):=\Big(\frac{\sin\pi z}{\pi z}\Big)^\nu P_{\nu-j-1}(\pi z)\frac{z^j}{j!}, \tag{2.23}$$

where $P_\ell(z)$ is the Taylor polynomial of degree ℓ of $(z/\sin z)^\nu$ about $z=0$.

Proposition 2.4.2. *Let $\nu\in\mathbb{N}$ and $0\le j<\nu$. The function $K_{\nu,j}(z)$ defined above is an entire functions in $L^2(\mathbb{R})$ of exponential type $\nu\pi$. It satisfies for any $\ell\in\{0,...,\nu-1\}$ and any $n\in\mathbb{Z}$*

$$K_{\nu,j}^{(\ell)}(n)=\begin{cases}1 & \text{if } j=\ell \quad \text{and } n=0\\ 0 & \text{else.}\end{cases} \tag{2.24}$$

Proof. Let ν and j as above. From Definition 2.4.1 we see that $K_{\nu,j}(z)$ is a square integrable, entire function of type $\nu\pi$. To prove (2.24), let $P(z)$ be any polynomial of degree smaller than ν. Then for any non-negative integer $\ell < \nu$ we have

$$\begin{aligned}\frac{d^\ell}{dz^\ell}\left[\left(\frac{\sin\pi z}{\pi z}\right)^\nu P(z)\right] &= \sum_{\mu=0}^{\ell}\binom{\ell}{\mu}\frac{d^\mu}{dz^\mu}\left[\left(\frac{\sin\pi z}{\pi z}\right)^\nu\right]P^{(\ell-\mu)}(z)\\ &= \frac{\sin\pi z}{\pi z}G(z)\end{aligned}$$

where $G(z)$ is some entire function, and so for any non-zero integer n and any $\ell < \nu$

$$\frac{d^\ell}{dz^\ell}\left[\left(\frac{\sin\pi z}{\pi z}\right)^\nu P(z)\right]\Bigg|_{z=n} = 0.$$

From the definition of the polynomial $P_{\nu-j-1}(z)$ we see that $K_{\nu,j}(z)$ is of the form

$$K_{\nu,j}(z) = \frac{z^j}{j!} + \sum_{m\geq\nu} c_m z^m.$$

This means that for $\ell < \nu$

$$K_{\nu,j}^{(\ell)}(0) = \begin{cases}1 & \text{if } j = \ell\\ 0 & \text{else.}\end{cases}$$

□

By the theorem of Paley and Wiener (Theorem 2.2.1), the Fourier transforms of the functions $K_{\nu,j}^{(\ell)}(z)$ are supported on $[-\nu/2, \nu/2]$. The next proposition shows that these Fourier transforms can be used to give an inverse of the Vandermonde matrix

$$A_\nu(t) := \left[(2\pi i(t+\mu+1-\nu))^\ell\right]_{\substack{0\leq\ell<\nu\\ 0\leq\mu<\nu}}, \tag{2.25}$$

where $t \in [\frac{\nu}{2}-1, \frac{\nu}{2}]$.

Proposition 2.4.3. *Let $\nu \in \mathbb{N}$, $j \in \{0,\nu\}$, and $t \in [\frac{\nu}{2}-1, \frac{\nu}{2}]$. The inverse matrix $A_\nu^{-1}(t) =: [b_{\mu,j}(t)]$ of the matrix $A_\nu(t)$ defined in (2.25) has entries*

$$b_{\mu,j}(t) = \widehat{K}_{\nu,j}(t+\mu+1-\nu). \tag{2.26}$$

Proof. For $t \in [\nu/2 - 1, \nu/2]$ and $0 \leq \ell, j < \nu$ define

$$u_{\ell,j}(t) := \sum_{\mu=0}^{\nu-1} [(2\pi i(t + \mu + 1 - \nu)]^{\ell} \widehat{K}_{\nu,j}(t + \mu + 1 - \nu) \tag{2.27}$$

and extend the $u_{\ell,j}$'s to $\mathbb{R}$ as 1–periodic functions. Performing ℓ differentiations in the representation

$$K_{\nu,j}(z) = \int_{-\nu/2}^{\nu/2} \widehat{K}_{\nu,j}(t) e(tz) dt$$

yields

$$K_{\nu,j}^{(\ell)}(z) = \int_{-\nu/2}^{\nu/2} (2\pi i t)^{\ell} \widehat{K}_{\nu,j}(t) e(tz) dt.$$

For $z = n \in \mathbb{Z}$ we obtain

$$\begin{aligned}
K_{\nu,j}^{(\ell)}(n) &= \int_{-\nu/2}^{\nu/2} (2\pi i t)^{\ell} \widehat{K}_{\nu,j}(t) e(tn) dt \\
&= \int_{\nu/2-1}^{\nu/2} \sum_{\mu=0}^{\nu-1} [(2\pi i(t + \mu + 1 - \nu)]^{\ell} \widehat{K}_{\nu,j}(t + \mu + 1 - \nu) \\
&\qquad\qquad\qquad\qquad \cdot e((t + \mu + 1 - \nu)n) dt \\
&= \int_{\nu/2-1}^{\nu/2} \sum_{\mu=0}^{\nu-1} [(2\pi i(t + \mu + 1 - \nu)]^{\ell} \widehat{K}_{\nu,j}(t + \mu + 1 - \nu) e(tn) dt \\
&= \int_{\nu/2-1}^{\nu/2} u_{\ell,j}(t) e(nt) dt,
\end{aligned}$$

and therefore $K_{\nu,j}^{(\ell)}(n)$ is the $-n$th Fourier coefficient of $u_{\ell,j}(t)$. Thus, from (2.24) we obtain

$$u_{j,\ell}(t) = \delta_{j,\ell}. \tag{2.28}$$

Here $\delta_{j,\ell}$ denotes the Kronecker symbol, which is equal to one if $j = \ell$ and zero otherwise.

Using (2.28), equation (2.27) can be written as the matrix equation

$$[\delta_{j,\ell}(t)]_{\substack{0 \leq \ell < \nu \\ 0 \leq j < \nu}} = A_{\nu}(t) [\widehat{K}_{\nu,j}(t + \mu + 1 - \nu)]_{\substack{0 \leq \mu < \nu \\ 0 \leq j < \nu}}.$$

So the inverse matrix of $A_\nu(t)$ has entries

$$b_{\mu,j}(t) = \widehat{K}_{\nu,j}(t+\mu+1-\nu).$$

$\square$

Cramer's rule together with the formula for the determinant of a Vandermonde matrix shows that the entries of $A_\nu^{-1}(t)$ are polynomials in t.

Now we are in a position to prove the generalization of Theorem 9 in [25].

Theorem 2.4.4. *Let $F(z)$ be an entire function in $L^2(\mathbb{R})$ of exponential type $\nu\pi$ for some $\nu \in \mathbb{N}$. Then the representation*

$$F(z) = \sum_{j=0}^{\nu-1} \sum_{k\in\mathbb{Z}} F^{(j)}(k) K_{\nu,j}(z-k) \tag{2.29}$$

is valid for all $z \in \mathbb{C}$.

Proof. For $t \in [\nu/2-1, \nu/2]$ and $0 \le \ell < \nu$ define

$$u_\ell(t) := \sum_{\mu=0}^{\nu-1} [(2\pi i(t+\mu+1-\nu)]^\ell \widehat{F}(t+\mu+1-\nu) \tag{2.30}$$

and extend the u_ℓ's to $\mathbb{R}$ as 1–periodic functions. We use the same method as in the proof of Proposition 2.4.3. Performing ℓ differentiations in the representation

$$F(z) = \int_{-\nu/2}^{\nu/2} \widehat{F}(t) e(tz)dt$$

yields

$$F^{(\ell)}(z) = \int_{-\nu/2}^{\nu/2} (2\pi it)^\ell \widehat{F}(t) e(tz)dt.$$

For $z = n \in \mathbb{Z}$ we obtain

$$\begin{aligned}
F^{(\ell)}(n) &= \int_{-\nu/2}^{\nu/2} (2\pi i t)^{\ell} \widehat{F}(t) e(tn)dt \\
&= \int_{\nu/2-1}^{\nu/2} \sum_{\mu=0}^{\nu-1} [(2\pi i(t+\mu+1-\nu)]^{\ell} \widehat{F}(t+\mu+1-\nu) \\
&\qquad\qquad\qquad\qquad \cdot e((t+\mu+1-\nu)n)dt \\
&= \int_{\nu/2-1}^{\nu/2} \sum_{\mu=0}^{\nu-1} [(2\pi i(t+\mu+1-\nu)]^{\ell} \widehat{F}(t+\mu+1-\nu) e(tn)dt,
\end{aligned}$$

and so $F^{(\ell)}(n)$ is the $-n$th Fourier coefficient of $u_\ell(t)$.

Equation (2.30) can be written as the equation $[u_\ell(t)]_\ell = A_\nu(t)[\widehat{F}(t+\mu+1-\nu)]_\mu$ with the matrix $A_\nu(t)$ defined in (2.25). Solving this equation using Proposition 2.4.3 we obtain

$$\big(\widehat{F}(t+\mu+1-\nu)\big)_{0\le\mu<\nu} = \big(b_{\mu,\ell}(t)\big)_{\substack{0\le\mu<\nu\\0\le\ell<\nu}} \big(u_\ell(t)\big)_{0\le\ell<\nu},$$

where $t \in [\nu/2-1, \nu/2]$. We obtain

$$\begin{aligned}
F(z) &= \int_{-\nu/2}^{\nu/2} \widehat{F}(t) e(tz)dt \qquad (2.31)\\
&= \sum_{\mu=0}^{\nu-1} \int_{-\nu/2+\mu}^{-\nu/2+\mu+1} \widehat{F}(t) e(zt)dt \\
&= \sum_{\mu=0}^{\nu-1} \int_{-\nu/2+\mu}^{-\nu/2+\mu+1} \sum_{\ell=0}^{\nu-1} b_{\mu,\ell}(t-\mu-1+\nu) u_\ell(t) e(zt)dt.
\end{aligned}$$

With

$$u_\ell(t,N) := \sum_{n=-N}^{N} F^{(\ell)}(n) e(-nt)$$

and $b_{\mu,\ell}(t-\mu-1+\nu)=\widehat{K}_{\nu,\ell}(t)$ we get

$$\begin{aligned}
&\sum_{\mu=0}^{\nu-1}\int_{-\nu/2+\mu}^{-\nu/2+\mu+1}\sum_{\ell=0}^{\nu-1} b_{\mu,\ell}(t-\mu-1+\nu)u_\ell(t,N)e(zt)dt \\
&\quad=\sum_{\mu=0}^{\nu-1}\int_{-\nu/2+\mu}^{-\nu/2+\mu+1}\sum_{\ell=0}^{\nu-1}\widehat{K}_{\nu,\ell}(t)\sum_{|n|\le N}F^{(\ell)}(n)e(-tn)e(zt) \\
&\quad=\sum_{\ell=0}^{\nu-1}\sum_{|n|\le N}F^{(\ell)}(n)\sum_{\mu=0}^{\nu-1}\int_{-\nu/2+\mu}^{-\nu/2+\mu+1}\widehat{K}_{\nu,\ell}(t)e(t(z-n))dt \\
&\quad=\sum_{\ell=0}^{\nu-1}\sum_{|n|\le N}\int_{-\nu/2}^{\nu/2}\widehat{K}_{\nu,\ell}(t)e(tz)dt \\
&\quad=\sum_{\ell=0}^{\nu-1}\sum_{|n|\le N}F^{(\ell)}(n)K_{\nu,\ell}(z-n). \qquad (2.32)
\end{aligned}$$

Since the sequences $F^{(j-1)}(n)$ are square summable by Lemma 2.2.3, the right hand side of the last equation converges uniformly on compact subsets of $\mathbb{C}$ as $N\to\infty$. On the left hand side we have that $u_\ell(t,N)$ converges to $u_\ell(t)$ as $N\to\infty$ in $L^2([0,1])$. Equations (2.31) and (2.32) finish the proof. $\square$

2.5 Construction of Majorants and Minorants

For $0\le\alpha\le 1$ define

$$F_\alpha(z):=\pi^{-2}\sin^2\pi(z-\alpha).$$

We set $\operatorname{sgn}_+(x):=\operatorname{sgn}(x+)$. The following definition provides us with the candidates for extremal minorants and majorants of $f(x)=\operatorname{sgn}(x)x^n$:

Definition 2.5.1. Define for $0\le\alpha\le 1$, $z\in\mathbb{C}$, and $n\in\mathbb{N}_0$

$$H_n(z;\alpha):=F_\alpha(z)\Big(z^n\sum_{k=-\infty}^{\infty}\frac{\operatorname{sgn}_+(k)}{(z-k-\alpha)^2}+2\sum_{k=1}^{n}B_{k-1}(\alpha)z^{n-k}+2\frac{B_n(\alpha)}{z-\{\alpha\}}\Big),$$

where $\{\alpha\}$ denotes the fractional part of α, and $B_n(\alpha)$ is the n-th Bernoulli polynomial evaluated at α (cf. Section 2.3). For $n=0$ the second sum is assigned the value zero.

We have the equality $B(z) = H_0(z;0)$, where $B(z)$ is Beurling's function defined in (1.2).

Note that $H_n(z;\alpha)$ is real entire, because the zeros of F_α cancel the poles of the first and the last term in parenthesis, and the second term is a real polynomial.

Lemma 2.5.2. *For any $n \in \mathbb{N}_0$ the function $H_n(\delta z;\alpha)$ is of type $2\pi\delta$.*

Proof. Recall that an entire function $G(z)$ is of type $2\pi\delta$, if for all $\varepsilon > 0$ there exists a constant A_ε such that $|G(z)| \leq A_\varepsilon e^{(2\pi\delta+\varepsilon)|z|}$ for $z \in \mathbb{C}$.

The expressions obtained by multiplying $F_\alpha(z)$ with the second and the third term in the parenthesis of Definition 2.5.1 are of type 2π. It remains to estimate the first term in Definition 2.5.1. The series $\sum_\ell |z-\ell-\alpha|^{-2} F_\alpha(z-\ell)$ is bounded uniformly for all z satisfying $|z-k-\alpha| < 1/4$ with some $k \in \mathbb{Z}$. Moreover, for all z and k satisfying $|z-k-\alpha| \geq 1/4$, the sum $\sum_\ell |z-\ell-\alpha|^{-2}$ is bounded uniformly in z. Since $F_\alpha(z)$ is of type 2π, it follows that $H_n(z;\alpha)$ is of type 2π as well, and we obtain that $H_n(\delta z;\alpha)$ is of type $2\pi\delta$. $\square$

We will see in (2.46) and (2.47) that the function $H_n(x;\alpha)$ is an extremal function for $\operatorname{sgn}(x)x^n$ precisely when the 1–periodic function $B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)$ has no changes of sign for all $t \in \mathbb{R}$. This motivates the following choices for the values of α.

Let $n \in \mathbb{N}$. By Lemma 2.3.2 the function $B_{2n}(t)$ $(n \geq 1)$ has exactly one zero z_{2n} in the interval $(0,1/2)$. Let $z_0 := 0$. (By the result of D. H. Lehmer [12] quoted in Section 2.3 we have $1/4 - \pi^{-1}2^{-2n-1} < z_{2n} < 1/4$ for $n \in \mathbb{N}$.) The odd Bernoulli polynomials $B_{2n+1}(t)$ have zeros at $t = 0$ and $t = 1/2$, but no zeros in the interval $(0,1/2)$ (cf. Section 2.3).

In the following definition α_n is a maximum and β_n is a minimum of $B_{n+1}(t)$ on $[0,1]$ (cf. Lemma 2.3.2):

$$\begin{array}{rclrcl} \alpha_{4k} & := & 1 - z_{4k}, & \beta_{4k} & := & z_{4k}, \\ \alpha_{4k+1} & := & 0, & \beta_{4k+1} & := & \frac{1}{2}, \\ \alpha_{4k+2} & := & z_{4k+2}, & \beta_{4k+2} & := & 1 - z_{4k+2}, \\ \alpha_{4k+3} & := & \frac{1}{2}, & \beta_{4k+3} & := & 0, \end{array} \tag{2.33}$$

where $k \in \mathbb{N}_0$.

With these definitions $\delta^{-n}H_n(\delta z;\alpha_n)$ and $\delta^{-n}H_n(\delta z;\beta_n)$ turn out to be the extremal minorant and the extremal majorant of type $2\pi\delta$ of $\mathrm{sgn}(x)x^n$, respectively:

Theorem 2.5.3. *Let $\delta > 0$ and $n \in \mathbb{N}_0$. The inequality*

$$\delta^{-n}H_n(\delta x;\alpha_n) \le \mathrm{sgn}(x)x^n \le \delta^{-n}H_n(\delta x;\beta_n) \tag{2.34}$$

holds for all $x \in \mathbb{R}$. Moreover,

(i) for every function $F \in E(2\pi\delta)$ satisfying $F(x) \ge \mathrm{sgn}(x)x^n$

$$\int_{\infty}^{\infty} \big(F(x) - \mathrm{sgn}(x)x^n\big)dx \ge -2\frac{B_{n+1}(\beta_n)}{(n+1)\delta^{n+1}} \tag{2.35}$$

with equality exactly for $F(x) = \delta^{-n}H_n(\delta x;\beta_n)$, and

(ii) for every function $G \in E(2\pi\delta)$ satisfying $G(x) \le \mathrm{sgn}(x)x^n$

$$\int_{\infty}^{\infty} \big(\mathrm{sgn}(x)x^n - G(x)\big)dx \ge 2\frac{B_{n+1}(\alpha_n)}{(n+1)\delta^{n+1}} \tag{2.36}$$

with equality exactly for $G(x) = \delta^{-n}H_n(\delta x;\alpha_n)$.

Note that α_n and β_n are independent of δ.

Proof. We will first deal with the case $\delta = 1$. At the end of this section we will indicate how to obtain majorants for any $\delta > 0$. Since most of the following statements are concerned with the difference of $H_n(x;\alpha)$ and $\mathrm{sgn}(x)x^n$ we define

$$\psi_{n,\alpha}(x) := H_n(x;\alpha) - \mathrm{sgn}(x)x^n. \tag{2.37}$$

The proof of Theorem 2.5.3 is divided into a series of lemmata.

Lemma 2.5.4. *Let $0 \le \alpha \le 1$ and $n \in \mathbb{N}_0$. The function $\psi_{n,\alpha}(x)$ ($x \in \mathbb{R}$) is absolutely integrable. Moreover, if $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ are defined by* (2.33)*, then*

$$H_n(x;\alpha_n) \le \mathrm{sgn}(x)x^n \le H_n(x;\beta_n).$$

Proof. Let $x \in \mathbb{R}$ and $0 \leq \alpha \leq 1$. We will consider the cases $x > 0$ and $x < 0$ separately. Let $\operatorname{sgn}_+(x)$ be the right-continuous version of $\operatorname{sgn}(x)$, i.e. $\operatorname{sgn}_+(x) := \operatorname{sgn}(x+)$. For $x > 0$ we have

$$\psi_{n,\alpha}(x) = H_n(x;\alpha) - x^n. \tag{2.38}$$

We write x^n in the form

$$x^n = x^n \Big(\frac{\sin \pi(x-\alpha)}{\pi}\Big)^2 \sum_{k=-\infty}^{\infty} \frac{1}{(x-k-\alpha)^2}. \tag{2.39}$$

This is a well-known partial fraction expansion of $1/\sin^2 \pi x$.

We compare the series occuring in (2.38) given by (2.39) and Definition 2.5.1. We see from Lemma 2.3.4 with $\mu = n+1$ that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}_+(k+\alpha)}{(x-k-\alpha)^2} + 2\sum_{\ell=1}^{n} \frac{B_{\ell-1}(\alpha)}{x^\ell} + \frac{2B_n(\alpha)}{x^n(x-\{\alpha\})} - \sum_{k=-\infty}^{\infty} \frac{1}{(x-k-\alpha)^2} \\
&\quad = -2\sum_{k=1}^{\infty} \frac{1}{(x+k-\alpha)^2} + 2\sum_{\ell=1}^{n+1} \frac{B_{\ell-1}(\alpha)}{x^\ell} + \frac{B_n(\alpha)}{x^n}\Big(\frac{2}{x-\{\alpha\}} - \frac{2}{x}\Big) \\
&\quad = -2(n+2)\int_0^\infty \frac{B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)}{(x+t)^{n+3}}\, dt + O\left(x^{-n-2}\right) \qquad (2.40) \\
&\quad \ll x^{-n-2},
\end{aligned}$$

because $B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)$ is bounded on $\mathbb{R}$. Therefore we obtain

$$\psi_{n,\alpha}(x) = H_n(x;\alpha) - x^n \ll x^{-2} \tag{2.41}$$

for $x > 0$.

Now let $x < 0$. In this region we have

$$\psi_{n,\alpha}(x) = H_n(x;\alpha) + x^n. \tag{2.42}$$

We utilize (2.39), let $y = -x > 0$, and use $B_\ell(\alpha) = (-1)^\ell B_\ell(1-\alpha)$ (cf.

(2.18)) to obtain

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}_+(k+\alpha)}{(x-k-\alpha)^2} + 2\sum_{\ell=1}^{n} \frac{B_{\ell-1}(\alpha)}{x^\ell} + \frac{2B_n(\alpha)}{x^n(x-\{\alpha\})} + \sum_{k=-\infty}^{\infty} \frac{1}{(x-k-\alpha)^2}\\
&\quad = 2\sum_{k=0}^{\infty} \frac{1}{(x-k-\alpha)^2} + 2\sum_{\ell=1}^{n} \frac{B_{\ell-1}(\alpha)}{x^\ell} + \frac{2B_n(\alpha)}{x^n(x-\{\alpha\})}\\
&\quad = 2\sum_{k=0}^{\infty} \frac{1}{(y+k+\alpha)^2} + 2\sum_{\ell=1}^{n} \frac{(-1)^\ell B_{\ell-1}(\alpha)}{y^\ell} - \frac{2(-1)^n B_n(\alpha)}{y^n(y+\{\alpha\})}\\
&\quad = 2\sum_{k=0}^{\infty} \frac{1}{(y+k+\alpha)^2} - 2\sum_{\ell=1}^{n+1} \frac{B_{\ell-1}(1-\alpha)}{y^\ell} + \frac{B_n(1-\alpha)}{y^n}\Big(\frac{2}{y} - \frac{2}{y+\{\alpha\}}\Big)\\
&\quad = 2(n+2)\int_0^\infty \frac{B_{n+1}(1-\alpha) - \mathcal{B}_{n+1}(t-\alpha)}{(y+t)^{n+3}} dt + O(y^{-n-2}) \qquad (2.43)\\
&\quad \ll y^{-n-2}.
\end{aligned}
$$

This implies for $x < 0$ that

$$\psi_{n,\alpha}(x) = H_n(x;\alpha) + x^n \ll x^{-2}. \tag{2.44}$$

(2.41) and (2.44) prove the first statement of Lemma 2.5.4.

For the second statement we use the representation for $\psi_{n,\alpha}(x)$ derived in (2.40) and (2.43). If

$$\frac{B_n(\alpha)}{x^n}\Big(\frac{2}{x-\{\alpha\}} - \frac{2}{x}\Big) = 0, \tag{2.45}$$

then (2.40) implies for $x > 0$

$$\psi_{n,\alpha}(x) = -2(n+2)F(x-\alpha)x^n \int_0^\infty \frac{B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)}{(x+t)^{n+3}}\, dt \tag{2.46}$$

and (2.43) implies for $x < 0$

$$\psi_{n,\alpha}(x) = 2(n+2)F(x-\alpha)(-x)^n \int_0^\infty \frac{B_{n+1}(1-\alpha) - \mathcal{B}_{n+1}(t+1-\alpha)}{(-x+t)^{n+3}}\, dt. \tag{2.47}$$

If $B_{n+1}(t)$ restricted to $[0,1]$ has a maximum at $t = \alpha$, then it has a minimum at $t = 1-\alpha$ if n is even, and a maximum if n is odd, since $B_\ell(\alpha) = (-1)^\ell B_\ell(1-\alpha)$. This implies that for such α the expressions $B_{n+1}(\alpha) -$

$\mathcal{B}_{n+1}(t+\alpha)$ and $B_{n+1}(1-\alpha) - \mathcal{B}_{n+1}(t+1-\alpha)$ do not change their signs for $t \in [0,\infty)$, and since $-x^n = (-x)^n(-1)^{n+1}$ we obtain that for such α the expressions in (2.46) and (2.47) are either both positive or both negative for all x in the respective ranges. Moreover, $\psi_{n,\alpha} \geq 0$ if $B_{n+1}(t)$ assumes its minimum on $[0,1]$ at $t=\alpha$, and $\psi_{n,\alpha} \leq 0$ if $B_{n+1}(t)$ assumes its maximum at $t=\alpha$.

Since by (2.33) and Lemma 2.3.2 the function $B_{n+1}(t)$ assumes a minimum on $[0,1]$ at $t=\beta_n$, and a maximum at $t=\alpha_n$ we have

$$H_n(x;\alpha_n) \leq \operatorname{sgn}(x)x^n \leq H_n(x;\beta_n),$$

and this finishes the proof of Lemma 2.5.4. $\square$

Next we will show that there is equality in (2.35) for $F(x) = H_n(x;\beta_n)$ and (2.36) for $G(x) = H_n(x;\alpha_n)$. To this end we will compute the Fourier transform $\widehat{\psi}_{n,\alpha}(t)$. Since $\psi_{n,\alpha}(x)$ is integrable, its Fourier transform exists. Its value is given by

Lemma 2.5.5. *Let $0 \leq \alpha \leq 1$ and $n \in \mathbb{N}_0$. We have*

$$\widehat{\psi}_{n,\alpha}(t) = -2\sum_{k=0}^{\infty}\frac{B_{k+n+1}(\alpha)}{(k+1)!}\Big(\frac{k+1}{k+n+1} - |t|\Big)(-2\pi i t)^k \qquad (2.48)$$

$$\qquad + \frac{B_n(\alpha)}{\pi i}\operatorname{sgn}(t)\big(e(-\{\alpha\}t) - 1\big) \textit{ for } |t| < 1,$$

$$\widehat{\psi}_{n,\alpha}(t) = -\frac{2\cdot n!}{(2\pi i t)^{n+1}} \textit{ for } |t| \geq 1. \qquad (2.49)$$

By taking the value of $\widehat{\psi}_{n,\alpha}(t)$ at $t=0$ in (2.48) we obtain the equalities in (2.35) for $F(x) = H_n(x;\beta_n)$ and in (2.36) for $G(x) = H_n(x;\alpha_n)$.

Proof of Lemma 2.5.5. Recall $\operatorname{sgn}_+(x) = \operatorname{sgn}(x+)$, and let

$$F(z) = \pi^{-2}\sin^2 \pi z \text{ for } z \in \mathbb{C}.$$

Performing the index shift $k+n+1 \mapsto k$ in the series representing $\widehat{\psi}_{n,\alpha}(t)$

for $|t| < 1$ leads to (2.48) in the form in which we will prove it:

$$\widehat{\psi}_{n,\alpha}(t) = -2\sum_{k=n+1}^{\infty}\frac{B_k(\alpha)}{(k-n)!}\Big(\frac{k-n}{k} - |t|\Big)(-2\pi it)^{k-n-1} + \frac{B_n(\alpha)}{\pi i}\operatorname{sgn}(t)\big(e(-\{\alpha\}t) - 1\big) \text{ for } |t| < 1. \quad (2.50)$$

We shall prove the lemma using an induction on n. To do this we will need the Fourier transform of $H_0'(x;\alpha)$. This will be done with a method similar to the proof of Theorem 6 in [25]. Define

$$H_{0,K}(x;\alpha) := F(x-\alpha)\Big(\sum_{k=-K}^{K-1}\frac{\operatorname{sgn}_+(k+\alpha)}{(x-k-\alpha)^2} + \frac{2}{x-\{\alpha\}}\Big)$$

and observe that $H_{0,K}(x;\alpha)$ converges uniformly to $H_0(x;\alpha)$ on compact subsets of $\mathbb{C}$. With the Fourier expansions

$$\frac{F(x)}{x^2} = \int_{-1}^{1}(1-|t|)e(xt)dt, \quad (2.51)$$

$$\frac{F(x)}{x} = \frac{1}{2\pi i}\int_{-1}^{1}\operatorname{sgn}(t)e(xt)dt, \quad (2.52)$$

and with $F(x-\alpha) = F(x-\{\alpha\})$ we obtain

$$H_{0,K}(x;\alpha) = \int_{-1}^{1}(1-|t|)\Big[\sum_{k=0}^{K-1}e(-(k+\alpha)t) - \sum_{k=-K}^{-1}e(-(k+\alpha)t)\Big]e(xt)dt + \frac{1}{\pi i}\int_{-1}^{1}\operatorname{sgn}(t)e(-\{\alpha\}t)e(xt)dt.$$

We have for $t \neq 0$ and $|t| < 1$

$$\sum_{k=0}^{K-1}e(-(k+\alpha)t) = e(-\alpha t)\frac{1-e(-Kt)}{1-e(-t)},$$

$$-\sum_{k=-K}^{-1}e(-(k+\alpha)t) = -e(-\alpha t)e(t)\frac{1-e(Kt)}{1-e(t)} = -e(-\alpha t)\frac{1-e(Kt)}{e(-t)-1},$$

and therefore

$$\sum_{k=0}^{K-1} e(-(k+\alpha)t) - \sum_{k=-K}^{-1} e(-(k+\alpha)t) = 2\frac{e(-\alpha t)}{1-e(-t)}(1-\cos 2\pi K t).$$

Since the last expression is bounded in a neighborhood of $t=0$ we obtain

$$\begin{aligned} H_{0,K}(x;\alpha) = \int_{-1}^{1} (1-|t|)\Big[\frac{2e(-\alpha t)}{1-e(-t)} - e(-\alpha t)\frac{2\cos 2\pi K t}{1-e(-t)}\Big] e(xt)dt \\ + \frac{1}{\pi i}\int_{-1}^{1} \operatorname{sgn}(t) e(-\{\alpha\}t) e(xt)dt. \end{aligned}$$

In order to apply the Lemma of Riemann - Lebesgue we have to remove the poles in the fractions of the first integral. We do this by differentiating both sides with respect to x and divide the resulting expression by 2. We obtain

$$\begin{aligned} \frac{1}{2}H'_{0,K}(x;\alpha) = \int_{-1}^{1} (1-|t|)\Big[\frac{2\pi i t\, e(-\alpha t)}{1-e(-t)} - e(-\alpha t)\frac{2\pi i t \cos 2\pi K t}{1-e(-t)}\Big] e(xt)dt \\ + \int_{-1}^{1} |t| e(-\{\alpha\}t) e(xt)dt. \quad (2.53) \end{aligned}$$

By the Lemma of Riemann - Lebesgue we have

$$\lim_{K\to\infty} \int_{-1}^{1} \frac{2\pi i t \cos 2\pi K t}{1-e(-t)} e(xt)dt = 0. \quad (2.54)$$

Since $\{H_{0,K}(x;\alpha)\}_{K\in\mathbb{N}}$ is a sequence of entire functions that converges to $H_0(x;\alpha)$ uniformly on any compact subset of $\mathbb{C}$, the sequence of derivatives $\{H'_{0,K}(x;\alpha)\}_{K\in\mathbb{N}}$ converges to $H'_0(x;\alpha)$ uniformly on any compact subset of $\mathbb{C}$. Thus (2.53) and (2.54) imply

$$\frac{1}{2}H'_0(x;\alpha) = \int_{-1}^{1} \Big[(1-|t|)\frac{2\pi i t\, e(-\alpha t)}{1-e(-t)} + |t| e(-\{\alpha\}t)\Big] e(xt)dt. \quad (2.55)$$

Equation (2.12) applied to $(-2\pi it)e(-\alpha t)(e(-t)-1)^{-1}$ for $|t|<1$ shows

$$\Big[\frac{1}{2}H_0'(x;\alpha)\Big]^{\wedge}(t) = (1-|t|)\sum_{k=0}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^k + |t|e(-\{\alpha\}t) \tag{2.56}$$
$$= 1+(1-|t|)\sum_{k=1}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^k + |t|\big(e(-\{\alpha\}t)-1\big)$$

for $|t|<1$. Equation (2.55) implies

$$\big[\frac{1}{2}H_0'(x;\alpha)\big]^{\wedge}(t) = 0 \text{ for } |t|\geq 1. \tag{2.57}$$

Now we can prove (2.49) and (2.50) by induction on n. The difference $\psi_{0,\alpha}(x) = H_0(x,\alpha)-\operatorname{sgn}(x)$ is absolutely integrable by Lemma 2.5.4, so its Fourier transform exists.

From

$$\frac{1}{2}\int_{-\infty}^{\infty} e(-xt)\,d\psi_{0,\alpha}(x) = \Big[\frac{1}{2}H_0'(x,\alpha)\Big]^{\wedge}(t)-1$$

we obtain with (2.56) and $2\pi it\widehat{f}(t)=\widehat{f'}(t)$ that for $|t|<1$

$$\begin{aligned}\widehat{\psi}_{0,\alpha}(t) &= \frac{1}{\pi it}\Big(\Big[\frac{1}{2}H_0'(x;\alpha)\Big]^{\wedge}(t)-1\Big)\\ &= \frac{1}{\pi it}\Big((1-|t|)\sum_{k=1}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^k + |t|\big(e(-\{\alpha\}t)-1\big)\Big)\\ &= -2(1-|t|)\sum_{k=1}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^{k-1} + \frac{\operatorname{sgn}(t)}{\pi i}\big(e(-\{\alpha\}t)-1\big),\end{aligned}$$

and this is (2.50) for $n=0$. Moreover, equation (2.57) implies for $|t|\geq 1$ that

$$\widehat{\psi}_{0,\alpha}(t) = \frac{1}{\pi it}\Big(\Big[\frac{1}{2}H_0'(x,\alpha)\Big]^{\wedge}(t)-1\Big) = -\frac{1}{\pi it},$$

and this is (2.49) for $n=0$.

Induction step. We start out by showing that the Fourier transforms $\widehat{\psi}_{n,\alpha}$ satisfy a recursion in n. Assume that (2.49) and (2.50) are true for some

$n \in \mathbb{N}_0$. From Definition 2.5.1 with n and $n+1$ we obtain

$$H_{n+1}(x;\alpha) = xH_n(x;\alpha) + 2F(x-\alpha)\,\frac{B_{n+1}(\alpha) - \{\alpha\}B_n(\alpha)}{x - \{\alpha\}}. \tag{2.58}$$

By (2.50), the Fourier transforms of $\psi_{n,\alpha}$ and $\psi_{n+1,\alpha}$ exist. Subtracting $\operatorname{sgn}(x)x^{n+1} = x\operatorname{sgn}(x)x^n$ on both sides of (2.58), taking Fourier transforms, and applying (2.52) to $(x-\{\alpha\})^{-1}F(x-\alpha)$ we obtain for $|t|<1$ the recursion

$$\widehat{\psi}_{n+1,\alpha}(t) = -\frac{1}{2\pi i}\frac{d}{dt}\widehat{\psi}_{n,\alpha}(t) + \frac{1}{\pi i}(B_{n+1}(\alpha) - \{\alpha\}B_n(\alpha))\operatorname{sgn}(t)e(-\{\alpha\}t). \tag{2.59}$$

By the induction hypothesis, (2.50) holds for n, i.e. for $|t|<1$

$$\begin{aligned}\widehat{\psi}_{n,\alpha}(t) = -2\sum_{k=n+1}^{\infty}\frac{B_k(\alpha)}{(k-n)!}\Big(\frac{k-n}{k} - |t|\Big)(-2\pi i t)^{k-n-1}& \\ + \frac{B_n(\alpha)}{\pi i}\operatorname{sgn}(t)\big(e(-\{\alpha\}t) - 1\big).&\end{aligned} \tag{2.60}$$

For $k \geq n+2$

$$\frac{d}{dt}\Big(\frac{k-n}{k} - |t|\Big)t^{k-n-1} = (k-n)\Big(\frac{k-n-1}{k} - |t|\Big)t^{k-n-2}. \tag{2.61}$$

Inserting (2.60) in (2.59), splitting off the first term of the series, and applying (2.61) to the remaining part of the series we obtain for $|t|<1$

$$\begin{aligned}\widehat{\psi}_{n+1,\alpha}(t) &= -2\sum_{k=n+2}^{\infty}\frac{B_k(\alpha)}{(k-n-1)!}\Big(\frac{k-n-1}{k} - |t|\Big)(-2\pi i t)^{k-n-2} \\ &\quad - \frac{B_{n+1}(\alpha)\operatorname{sgn}(t)}{\pi i} + \frac{\operatorname{sgn}(t)B_n(\alpha)}{(-2\pi i)\pi i}(-2\pi i\{\alpha\})e(-\{\alpha\}t) \\ &\quad + \frac{1}{\pi i}\big(B_{n+1}(\alpha) - \{\alpha\}B_n(\alpha)\big)\operatorname{sgn}(t)e(-\{\alpha\}t) \\ &= -2\sum_{k=n+2}^{\infty}\frac{B_k(\alpha)}{(k-n-1)!}\Big(\frac{k-n-1}{k} - |t|\Big)(-2\pi i t)^{k-n-2} \\ &\qquad + \frac{B_{n+1}(\alpha)}{\pi i}\operatorname{sgn}(t)(e(-\{\alpha\}t) - 1),\end{aligned}$$

and this is (2.50) for $n+1$.

Since the Fourier transform of $(x-\{\alpha\})^{-1}\sin^2\pi(x-\alpha)$ equals zero outside the interval $[-1,1]$, we have with (2.58) for $|t| \geq 1$

$$\widehat{\psi}_{n+1,\alpha}(t) = -\frac{1}{2\pi i}\frac{d}{dt}\widehat{\psi}_{n,\alpha}(t) = -\frac{2(n+1)!}{(2\pi i t)^{n+2}},$$

and this is (2.49) for $n+1$. $\square$

The next lemma establishes the extremality properties of $H_n(x;\alpha)$.

Lemma 2.5.6. *Let $n \in \mathbb{N}_0$, and let $F_n, G_n \in E(2\pi)$ be real entire functions such that*

$$G_n(x) \leq \operatorname{sgn}(x)x^n \leq F_n(x)$$

for all $x \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty}(F_n(x) - \operatorname{sgn}(x)x^n)dx \geq -\frac{2}{n+1}\min_{0\leq t\leq 1} B_{n+1}(t), \tag{2.62}$$

$$\int_{-\infty}^{\infty}(\operatorname{sgn}(x)x^n - G_n(x))dx \geq \frac{2}{n+1}\max_{0\leq t\leq 1} B_{n+1}(t). \tag{2.63}$$

Moreover, in (2.62) *and* (2.63) *equality can hold only for the minorants and majorants defined in Lemma 2.5.4.*

Proof. Let $0 \leq \alpha \leq 1$, and let $F_n \in E(2\pi)$ be a majorant for $\operatorname{sgn}(x)x^n$. Assume that

$$\int_{-\infty}^{\infty}(F_n(x) - \operatorname{sgn}(x)x^n)dx < \infty. \tag{2.64}$$

Let $\psi_n(x) = F_n(x) - \operatorname{sgn}(x)x^n$, and recall that $\psi_{n,\alpha}(x) = H_n(x;\alpha) - \operatorname{sgn}(x)x^n$. By (2.64) and Lemma 2.5.4 the difference $F_n(x) - H_n(x;\alpha) = \psi_n(x) - \psi_{n,\alpha}(x)$ is absolutely integrable and an entire function of type 2π. Using the Paley-Wiener Theorem (Theorem 2.2.1), we see that the support of the Fourier transform of $F_n(x) - H_n(x;\alpha)$ is a subset of $[-1,1]$, i.e.

$$[F_n - H_n(\cdot;\alpha)]^\wedge(t) = 0 \text{ for } |t| \geq 1.$$

It follows from equation (2.49) that

$$\widehat{\psi}_n(t) = \widehat{\psi}_{n,\alpha}(t) = -\frac{2n!}{(2\pi i t)^{n+1}} \text{ for } |t| \geq 1. \tag{2.65}$$

Now use (2.65), the Poisson summation formula (cf. Theorem 2.66), and the Fourier series expansion of the Bernoulli periodic functions given in Lemma 2.3.1 to obtain that

$$0 \leq \sum_{\ell=-\infty}^{\infty} \psi_n(\ell+t) = \widehat{\psi}_n(0) - \lim_{\varepsilon\to 0+} \frac{2}{n+1} \sum_{k\neq 0} \widehat{K}(\varepsilon k) \frac{(n+1)!}{(2\pi i k)^{n+1}} e(kt) \quad (2.66)$$
$$= \widehat{\psi}_n(0) + \frac{2}{n+1} \mathcal{B}_{n+1}(t).$$

Since this has to hold for all $t \in [0,1]$,

$$\widehat{\psi}_n(0) \geq -\frac{2}{n+1} \min_{0\leq t\leq 1} B_{n+1}(t), \quad (2.67)$$

which proves (2.62).

Similarly, inequality (2.63) follows with $\phi_n(x) = \operatorname{sgn}(x)x^n - G_n(x)$ and

$$\widehat{\phi}_n(0) \geq \frac{2}{n+1} \max_{0\leq t\leq 1} B_{n+1}(t). \quad (2.68)$$

It remains to investigate when we have equality in (2.62) and (2.63). Let $0 \leq \alpha \leq 1$ such that $B_{n+1}(t)$ has its minimum on $[0,1]$ at $t = \alpha$. If $F_n \in E(2\pi)$ is chosen such that F_n is a majorant of $\operatorname{sgn}(x)x^n$ with

$$\widehat{\psi}_n(0) = -\frac{2}{n+1} B_{n+1}(\alpha),$$

then we have equality in (2.66) for $t = \alpha$. This means that $\psi_n(\ell+\alpha) = 0$ for all integers ℓ, i.e.

$$F(\alpha+k) = \operatorname{sgn}_+(\alpha+k)(\alpha+k)^n \text{ for all } k \in \mathbb{Z}.$$

The same reasoning applies to $H_n(x;\alpha)$. If $\alpha = 0$ or 1, let $n \geq 2$. Since both $F_n(x)$ and $H_n(x;\alpha)$ are majorants of $\operatorname{sgn}(x)x^n$, they must have the same derivatives at the numbers $\alpha + k$, namely $n \cdot \operatorname{sgn}(\alpha+k)(\alpha+k)^{n-1}$. From Theorem 2.4.4 for $\nu = 2$ we obtain

$$F_n(z) - H_n(z;\alpha) = 0$$

for all $z \in \mathbb{C}$.

If $n \in \{0,1\}$ and $\alpha \in \{0,1\}$, we cannot immediately conclude that $F_n(x)$

and $H_n(x;\alpha)$ have equal derivatives at $x = 0$. However, as in the proof of Theorem 8 in [25], an application of Theorem 2.4.4 with $\nu = 2$ implies

$$F_n(z) - H_n(z;\alpha) = (F_n'(0) - H_n'(0;\alpha))\pi^{-2}x^{-1}\sin^2 \pi z.$$

The function $F - H_n(.;\alpha)$ is integrable as shown above. Since $x^{-1}\sin^2 \pi z$ is not integrable, we must have $F_n'(0) - H_n'(0;\alpha) = 0$. Thus, $F_n(z) = H_n(z;\alpha)$ holds in this case as well. The computation for $G_n(z)$ goes along the same lines. □

It remains to prove Theorem 2.5.3 for arbitrary $\delta > 0$. Defining $f_\delta(x) := f(\delta x)$ for any $f \in L^2(\mathbb{R})$, we note that $\widehat{f_\delta}(t) = \delta^{-1}\widehat{f}(t/\delta)$. Moreover, from Lemma 2.5.4 we obtain

$$H_n(\delta x;\alpha_n) \leq \operatorname{sgn}(\delta x)(\delta x)^n \leq H_n(\delta x;\beta_n),$$

and therefore with $\operatorname{sgn}(\delta x) = \operatorname{sgn}(x)$

$$\delta^{-n}H_n(\delta x;\alpha_n) \leq \operatorname{sgn}(x)x^n \leq \delta^{-n}H_n(\delta x;\beta_n).$$

Defining $f_\delta(x) := f(\delta x)$ for any $f \in L^2(\mathbb{R})$ we note that $\widehat{f_\delta}(t) = \delta^{-1}\widehat{f}(t/\delta)$. This implies

$$\int_{-\infty}^{\infty} (\delta^{-n}H_n(\delta x;\alpha) - \operatorname{sgn}(x)x^n)dx = \delta^{-n}\widehat{\psi}_{n,\alpha}(0) = \frac{2}{(n+1)\delta^{n+1}}B_{n+1}(\alpha).$$

This completes the proof of Theorem 2.5.3. □

Chapter 3

Extremal Functions via Laplace Transforms

3.1 Introduction

The representation of the functions $H_n(z;\alpha)$ used in Chapter 2 allows an elementary approach to the problem of majorizing $\mathrm{sgn}(x)x^n$, but it does not seem to be possible to use a similar representation for the problem of majorizing $\mathrm{sgn}(x)P(x)$. In this chapter a different approach to the problem of majorizing $\mathrm{sgn}(x)x^n$ is given. We shall show that this method gives entire majorants and minorants for functions of the form $\mathrm{sgn}(x)(ax+b)$ with real a and b.

As mentioned already in Chapter 1, the Laplace transform can be used to give entire approximations on the real line to $\mathrm{sgn}(x)P(x)$. Here the two-sided Laplace transform is defined by

$$\mathcal{L}[g](z) := \int_{-\infty}^{\infty} g(t)e^{-zt}dt \tag{3.1}$$

for all $z \in \mathbb{C}$ for which this integral converges absolutely.

The approximations to $\mathrm{sgn}(x)P(x)$ are obtained in the following way. For an entire function $F(z)$ satisfying $F^{-1}(z) = \mathcal{L}[g](z)$ with some measurable $g : \mathbb{R} \to \mathbb{R}$ in a vertical strip $\mathfrak{a} < \Re z < \mathfrak{b}$ in $\mathbb{C}$, we define

$$G_{F,P}(z) := -\frac{F(z)}{z}\mathcal{L}[\mathrm{sgn} \cdot P(D)g'](z), \tag{3.2}$$

where $D := d/dt$ denotes the differential operator.

The integral in (3.2) converges only in the strip $\mathfrak{a} < \Re z < \mathfrak{b}$ but we show in Theorem 3.5.2 that it extends under suitable conditions to an analytic function in the entire complex plane. Since we are interested in obtaining entire majorants and minorants for $\operatorname{sgn}(x)P(z)$ on the real line, it is necessary to investigate the sign changes of $\mathcal{L}[\operatorname{sgn} \cdot P(D)g'](z)$ on the real axis. We manage to do so (for certain polynomials P at least) by restricting the pairs F and g that we consider.

The functions $F(z)$ are taken from the class of Pólya-Laguerre entire functions (defined in the next section). These functions are entire functions that are real on the real line and have only real zeros. They arise as uniform limits of polynomials with real roots only.

In Section 3.2 we present some classical results about Pólya-Laguerre entire functions. We indicate why the reciprocal of such a function F can be represented as a two-sided Laplace transform of a real valued function g in any vertical strip between two consecutive zeros of F, and we summarize some of the known growth estimates for the derivatives of g. The contents of this section can be found in various places in the literature. Our presentation follows [6] and [19].

In Section 3.3 we investigate the structure of the functions g more closely. We summarize a result by I. J. Schoenberg [19], which states that any integrable g as above is variation diminishing, i.e. for any continuous, bounded, real-valued function φ, the convolution $g * \varphi$ has no more sign changes than φ. We conclude Section 3.3 by showing that the variation diminishing property implies that $g + P$ has at most $\deg(P) + 2$ real zeros for any real polynomial P.

In Section 3.5 we investigate $G_{F,P}$ defined by (3.2). The estimate for the real zeros of $g + P$ given in Section 3.3 turns out to be a crucial element for the investigation of the sign changes of $\mathcal{L}[\operatorname{sgn} P(D)g'](z)$ on the real line.

In Section 3.6 we determine extremal entire majorants and minorants of $\operatorname{sgn}(x)P(z)$ on the real line for certain classes of polynomials.

3.2 Pólya Entire Functions

In this section we introduce Pólya-Laguerre entire functions. The presentation follows [6], [9], and [19].

Let $z = x + iy$ where $x, y \in \mathbb{R}$. The elementary example $\int_{-\infty}^{0} e^{(1-z)t} dt =$

$(1-z)^{-1}$, valid in $\Re z < 1$, can be written as

$$(1-z)^{-1} = \mathcal{L}[\mathfrak{g}](z) \quad (\Re z < 1), \tag{3.3}$$

where $\mathfrak{g}(t) = e^t$ for $t < 0$, $\mathfrak{g}(0) = 1/2$, and $\mathfrak{g}(t) = 0$ for $t > 0$, and $\mathcal{L}$ denotes the bilateral Laplace transform introduced in (3.1).

Given two Laplace integrals $A(z) = \mathcal{L}[f](z)$ and $B(z) = \mathcal{L}[g](z)$ which are absolutely convergent in a vertical strip which includes the origin, Fubini's theorem implies that

$$A(z)B(z) = \mathcal{L}[f * g](z), \tag{3.4}$$

where $f * g(t) = \int_{\mathbb{R}} f(u)g(t-u)du$ denotes the integral convolution of f and g. Define for any nonzero real number a an operator T_a on $f : \mathbb{R} \to \mathbb{R}$ by

$$T_a f(t) := |a| f(at) \text{ for all } t \in \mathbb{R}.$$

The principle of induction applied to equations (3.4) and (3.3) implies

Lemma 3.2.1. *Let $P(z) = (1 - z/a_1)...(1 - z/a_n)$ with $a_i \in \mathbb{R}\backslash\{0\}$ for all i. Then*

$$P^{-1}(z) = \mathcal{L}[g](z) \text{ in } \mathfrak{a} < \Re z < \mathfrak{b}, \tag{3.5}$$

where $g = T_{a_1}\mathfrak{g} * ... * T_{a_n}\mathfrak{g}$, the number $\mathfrak{a}$ is the largest negative zero of $P(z)$ (and $\mathfrak{a} = -\infty$ if no such zero exists), and $\mathfrak{b}$ is the smallest positive zero of $P(z)$ (and $\mathfrak{b} = \infty$ if no such zero exists). In particular, $g(t)$ is non-negative on $\mathbb{R}$.

In order to determine the closure of the set of polynomials with real roots under uniform limits we define a class of functions introduced by Laguerre [11].

Definition 3.2.2. The class $\mathcal{E}$ of Laguerre-Pólya entire functions consists of all entire functions of the form

$$F(z) = C \exp(-cz^2 + bz) z^\kappa \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp(z/a_k), \tag{3.6}$$

where $c \geq 0$, $\kappa \in \mathbb{N}_0$, b, $a_k (k \in \mathbb{N})$ and C are real, and

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty. \tag{3.7}$$

From this definition it is clear that any $F \in \mathcal{E}$ is entire, real-valued on the real line, and has only real zeros. The elements of $\mathcal{E}$ are sometimes called Pólya-Laguerre entire functions.

A classical theorem by Laguerre [11] asserts that the functions in $\mathcal{E}$ are the uniform limits of polynomials having only real roots:

Theorem 3.2.3. *(a) If $F \in \mathcal{E}$, then there exists a sequence of polynomials P_n having only real roots such that for every $A > 0$*

$$\lim_{n\to\infty} P_n(z) = F(z)$$

uniformly in the circle $|z| \leq A$.

(b) If P_n is a sequence of polynomials such that

$$\lim_{n\to\infty} P_n(z) = F(z)$$

uniformly in some circle $|z| \leq A > 0$, then $F \in \mathcal{E}$.

It turns out that not only the reciprocals of polynomials with real roots, but the reciprocals of all non-constant elements in $\mathcal{E}$, have representations as Laplace transforms.

Theorem 3.2.4. *Let $F(z) \in \mathcal{E}$ with $F(0) \neq 0$, i.e. $\kappa = 0$ in (3.6).*

(a) If $F(z)$ is given by $F(z) := Ce^{bz}$ with $b, C \in \mathbb{R}$, then

$$F^{-1}(z) = C^{-1}e^{-bz} = C^{-1}\int_{-\infty}^{\infty} e^{-tz}\delta_b(dt),$$

where δ_b is the Dirac-measure at b.

(b) If $F(z)$ is not of the form Ce^{bz} and $F(0) \neq 0$, then there exists an integrable function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{F(z)} = \int_{-\infty}^{\infty} e^{-zt} g(t)dt \tag{3.8}$$

in a vertical strip containing the origin. If P_n is a sequence of polynomials with real roots converging to F uniformly on compact sets, then the corresponding sequence of functions given by (3.5) *converges pointwise to g. Moreover,*

$$g(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{zt}}{F(z)} dz. \tag{3.9}$$

(*If F is of the form $Ce^{bz}(1 - z/a)$ for real values a, b with $a \neq 0$, then* (3.9) *is understood to be the Cauchy principal value.*)

We omit the proofs of Theorems 3.2.3 and 3.2.4, since they are too long to reproduce here. We refer to [6], Chapters III and IV, and to [9], Chapter 7.

Remark 3.2.5. If P_n converges uniformly in a circle around the origin to $F \in \mathcal{E}$ with $F(0) \neq 0$, then for sufficiently large n the value of $P_n(z)$ at the origin will have the same sign as the value of $F(z)$ at the origin. Since g is a pointwise limit of functions of the form (3.5) multiplied by $F(0)$, the function g in (3.8) will be non-negative or non-positive, according to the sign of $F(0)$.

So far we have a representation for the reciprocal of $F \in \mathcal{E}$ only in a vertical strip containing the origin, provided that $F(0) \neq 0$. However, as we shall see in the next remark, these restrictions are not crucial. Define

$$S_F := \{b \in \mathbb{R} : F(b) = 0\} \cup \{\pm\infty\}. \tag{3.10}$$

Remark 3.2.6. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two consecutive elements in S_F. Choose a real number $\mathfrak{c}$ such that $\mathfrak{a} < \mathfrak{c} < \mathfrak{b}$. By applying Theorem 3.2.4 to $F(z - \mathfrak{c})$ we see that the reciprocal of any $F \in \mathcal{E}$ has a representation as a Laplace integral of a function $g : \mathbb{R} \to \mathbb{R}$ in the strip $\mathfrak{a} < \Re z < \mathfrak{b}$, provided that F is not of the form Ce^{bz}. Moreover, the function g can be represented as a product $g(t) = e^{\mathfrak{c}t} h(t)$, where h is integrable and by the previous remark non-negative or non-positive according to the sign of F in $\mathfrak{a} < \Re z < \mathfrak{b}$. Therefore, $g(t)$ is non-positive or non-negative according to the sign of F in this strip.

Next we want to determine the differentiability properties of the functions g. In order to do this we need some information about the decay of $F \in \mathcal{E}$

as a function of its imaginary part. The following lemma is based on [6], Theorem III.5.3.

Lemma 3.2.7. *Let $z = x + iy$ with $x, y \in \mathbb{R}$, and let $F \in \mathcal{E}$.*

(a) If $F(z)$ is of the form $e^{bz}P(z)$ for some polynomial P, we have for any $R > 0$

$$F^{-1}(x+iy) \ll (1+|y|)^{-\deg P} \tag{3.11}$$

as $|y| \to \infty$, uniformly in $|x| \leq R$.

(b) If $F(z)$ is not of the form $e^{bz}P(z)$ for some polynomial P, then for any $p > 0$ and $R > 0$

$$F^{-1}(x+iy) \ll (1+|y|)^{-p} \tag{3.12}$$

uniformly in the strip $|x| \leq R$ as $|y| \to \infty$.

Proof. Part (a) follows from $F^{-1}(z) \ll |P^{-1}(z)|$. For part (b) we have to investigate two cases. Let $p > 0$ be an arbitrary fixed number.

If F has at most a finite number of zeros and the factor e^{-cz^2} in 3.6 is non-trivial (i.e. $c > 0$), then for given $R > 0$ the estimate

$$F^{-1}(x+iy) \ll e^{-cy^2} \ll (1+|y|)^{-p}$$

holds in $|x| \leq R$ as $|y| \to \infty$.

If F has infinitely many zeros $\{a_k\}$, then choose N so large that $N > p$ and all zeros a_k with $k \geq N$ satisfy $|a_k| > R$. (This is possible since F is entire and not identically zero, i.e. the zeros of F have no limit point.) Under these conditions the tail

$$F_N(z) := \prod_{k=N}^{\infty} \Big(1 - \frac{z}{a_k}\Big) e^{z/a_k}$$

satisfies $|F_N| \geq \delta > 0$ in $|x| \leq R$ for some $\delta > 0$, and we obtain that

$$F^{-1}(x+iy) \ll |y|^{-N} \ll (1+|y|)^{-p}$$

for $|x| \leq R$ as $|y| \to \infty$. $\square$

Using this result we are in a position to investigate the Laplace inverse transformations of the reciprocals of $F \in \mathcal{E}$.

Lemma 3.2.8. *Let $F \in \mathcal{E}$, and let g be defined by*

$$F^{-1}(z) = \mathcal{L}[g](z)$$

in a vertical strip $\mathfrak{a} < \Re z < \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are two consecutive elements in S_F defined in (3.10)*. We have to consider two cases.*

(a) Let $F(z) = Ce^{bz}P(z)$ where P is a polynomial of degree $n \in \mathbb{N}$. If $n = 1$, then g has a discontinuity at $-b \in \mathbb{R}$. If $n \geq 2$, then g is $n-2$ times continuously differentiable.

(b) Assume that in (3.6) *either $c > 0$ or the product is infinite. Then the function g is infinitely differentiable.*

Proof. Let $F^{-1}(z) = \mathcal{L}[g](z)$ in the strip $\mathfrak{a} < \Re z < \mathfrak{b}$. By considering $F(z - \mathfrak{c})$ with some $\mathfrak{c} \in (\mathfrak{a}, \mathfrak{b})$ we can translate the origin inside the strip. Since $F(z-\mathfrak{c}) = \int_{\mathbb{R}} e^{-zt}e^{\mathfrak{c}t}g(t)dt$, this translation corresponds to multiplying g with the factor $e^{\mathfrak{c}t}$.

If $F(z) = C(1 - z/a)$ where $a \in \mathbb{R}\backslash\{0\}$, then $g(t) = C|a|\mathfrak{g}(at)$, where $\mathfrak{g}$ defined in (3.3) has a discontinuity at the origin. Since $e^{bz}F(z) = \int_{\mathbb{R}} e^{-zt}g(t+b)dt$, multiplication of $F(z)$ by e^{bz} corresponds to replacing $g(t)$ by $g(t+b)$.

If $F(z) = P(z)$, where P is a polynomial of degree ≥ 2 with $P(0) = 1$, then the integral in (3.9),

$$g(t) = \frac{1}{2\pi i}\int_{-i\infty}^{\infty} \frac{e^{zt}}{F(z)}dz, \tag{3.13}$$

can be differentiated at most $(n-2)$ times under the integral sign, and the resulting representation of $g^{(k)}(t)$ is continuous for $k = 0, ..., n-2$.

If the representation (3.6) contains a non-trivial factor of the form e^{-cz^2}, or if the product in (3.6) is infinite, then the representation (3.13) can be differentiated infinitely often under the integral by Lemma 3.11 (b). $\square$

In fact, more is true: it can be shown that in case (a) of Lemma 3.2.8 the $(n-2)$nd derivative is not continuously differentiable (cf. [6], proof of Theorem II.6.3).

We need an estimate of the rate of decay as $t \to \pm\infty$ of the derivatives introduced in the previous lemma. The statements of the following lemma are contained in [6], Theorem V.2.1.

Lemma 3.2.9. *Let $F \in \mathcal{E}$ have at least one zero. Let $\mathfrak{a} < \mathfrak{b}$ be two consecutive elements in S_F (defined in (3.10)), and let*

$$\frac{1}{F(z)} = \int_{-\infty}^{\infty} g(t)e^{-zt}dt \text{ for } \mathfrak{a} < \Re z < \mathfrak{b}. \tag{3.14}$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are both finite, then for any $n \in \mathbb{N}_0$ there are polynomials P_n and Q_n such that

$$g^{(n)}(t) \ll |P_n(t)|e^{\mathfrak{a}t} \text{ as } t \to \infty,$$
$$g^{(n)}(t) \ll |Q_n(t)|e^{\mathfrak{b}t} \text{ as } t \to -\infty.$$

If $\mathfrak{a} = -\infty$, then $g^{(n)}(t) \ll e^{-Kt}$ for all $K > 0$ as $t \to \infty$, and if $\mathfrak{b} = \infty$, then $g^{(n)}(t) \ll e^{Kt}$ for all $K > 0$ as $t \to -\infty$.

In particular, if (3.14) is valid in a neighborhood of the origin, $g^{(n)}(t)$ is integrable for any $n \in \mathbb{N}_0$.

Proof. (cf. proof of [6], Theorem V.2.1) If $F(z) = Ce^{bz}(1 - z/a)$ for $a, b \in \mathbb{R}$ with $a \neq 0$, then the statement follows from the representation

$$g(t) = C|a|\mathfrak{g}(a(t + b))$$

and the fact that $\mathfrak{g}(t) = e^t$ as $t \to -\infty$ and $\mathfrak{g}(t) = 0$ for $t > 0$.

Now assume that F has at $n \geq 2$ zeros. Let $\mathfrak{a}, \mathfrak{b} \in S_F$ such that $\mathfrak{a} < 0 < \mathfrak{b}$. (For the general case, consider $F(z - \mathfrak{c})$, where $\mathfrak{a} < \mathfrak{c} < \mathfrak{b}$.)

Let $0 \leq k \leq n - 2$. Since F has $n \geq 2$ zeros, the integral in

$$g^{(k)}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{z^k e^{zt}}{F(z)} dz$$

is absolutely convergent by Lemma 3.2.7. If $\mathfrak{b} < \infty$, let K be a number with $K \in (\mathfrak{b}, \mathfrak{b}_0)$ where $\mathfrak{b} < \mathfrak{b}_0$ are consecutive elements in S_F , and let $T > 0$ be large. Consider the rectangular contour Γ with endpoints $\pm iT$ and $K \pm iT$ traced clockwise. By Lemma 3.2.7 the integrals over top and bottom of the rectangle converge to zero as $T \to \infty$, so we obtain by the residue theorem

$$g(t) = -\operatorname{Res}\Big(\frac{e^{zt}}{F(z)}, z = \mathfrak{b}\Big) - \frac{1}{2\pi i}\int_{K-i\infty}^{K+i\infty} \frac{e^{zt}}{F(z)}dz,$$

and after differentiating this representation k times with respect to t we see that

$$g^{(k)}(t) = -\frac{d^k}{dt^k}\operatorname{Res}\Big(\frac{e^{zt}}{F(z)}, z = \mathfrak{b}\Big) - \frac{1}{2\pi i}\int_{K-i\infty}^{K+i\infty} \frac{z^k e^{zt}}{F(z)}dz. \tag{3.15}$$

We write $e^{zt} = e^{\mathfrak{b}t}e^{t(z-\mathfrak{b})}$, expand $e^{t(z-\mathfrak{b})}F^{-1}(z)$ in a Laurent series at $z = \mathfrak{b}$, and note that the residue of $e^{t(z-\mathfrak{b})}F^{-1}(z)$ is a polynomial P in t whose degree is one less than the multiplicity of the zero $\mathfrak{b}$ of F. As $t \to -\infty$ we have

$$\Big|\frac{1}{2\pi i}\int_{K-i\infty}^{K+i\infty} \frac{z^k e^{zt}}{F(z)}dz\Big| \ll e^{Kt}\int_{K-i\infty}^{K+i\infty} \frac{|x+iy|^k}{|F(x+iy)|}dy \ll e^{Kt}$$

(recall $k \le n-2$ and $F(x+iy) \ll |y|^{-n}$ in $|x| \le K$ by Lemma 3.2.7), and so

$$g^{(k)}(t) = \frac{d^k}{dt^k}e^{\mathfrak{b}t}P(t) + \mathcal{O}(e^{Kt})$$

as $t \to -\infty$. Since $K > \mathfrak{b}$, the first statement of the lemma follows. The remaining parts of the lemma are proved in a similar way. If $\mathfrak{a}$ or $\mathfrak{b}$ are infinite, we can choose any large K (positive or negative, as needed). The residue in (3.15) is not present in this case. $\square$

We need the representation of $\pi^2 \sin^{-2}\pi(x-\mathfrak{a})$ as a Laplace transform. The following way to derive this representation can be found in [6], Chapter III.9.

We start by noting that $1/\Gamma(1-z) \in \mathcal{E}$. With $g(t) := e^{-e^t}e^t$ we have

$$\int_{-\infty}^{\infty} e^{-zt}g(t)dt = \int_0^{\infty} e^{-t}t^{-z}dt = \Gamma(1-z) \text{ for } -\infty < \Re z < 1. \tag{3.16}$$

Since $(\pi z)^{-1}\sin \pi z = \prod_{k=1}^{\infty}(1 - z^2/k^2) = \{\Gamma(1+z)\Gamma(1-z)\}^{-1}$ we obtain with $g_-(t) := g(-t)$ that $\pi z \sin^{-1}\pi z = \int_{-\infty}^{\infty} g * g_-(t)e^{-zt}dt$ for $-1 < \Re z < 1$. The convolution in this representation can be computed directly:

$$g * g_-(x) = \int_{-\infty}^{\infty} e^{-e^{x-t}}e^{x-t}e^{-e^{-t}}e^{-t}dt = \frac{e^x}{(e^x+1)^2},$$

that is, we have

$$\frac{\pi z}{\sin \pi z} = \int_{-\infty}^{\infty} e^{-zt} \frac{e^t}{(e^t+1)^2} dt \text{ for } -1 < \Re z < 1. \tag{3.17}$$

Integrating by parts in the previous equation for $-1 < \Re z < 0$ we obtain $\pi z \sin^{-1} \pi z = -z \int_{-\infty}^{\infty} e^{-zt}(e^t+1)^{-1} dt$. It follows for $\alpha \in \mathbb{R}$ and $\alpha - 1 < \Re z < \alpha$ that

$$\frac{\pi}{\sin \pi(z-\alpha)} = -\int_{-\infty}^{\infty} e^{-zt} \frac{e^{\alpha t}}{e^t+1} dt. \tag{3.18}$$

After making the change of variable $u = e^t$ and a straightforward partial fraction decomposition we obtain

$$\int_{-\infty}^{\infty} \frac{e^{\alpha(x-t)}}{e^{x-t}+1} \frac{e^{\alpha t}}{e^t+1} dt = \frac{x e^{\alpha x}}{e^x - 1},$$

and thus for $\alpha \in \mathbb{R}$

$$\left(\frac{\pi}{\sin \pi(z-\alpha)}\right)^2 = \int_{-\infty}^{\infty} e^{-zt} \frac{t e^{\alpha t}}{e^t - 1} dt \text{ where } \alpha - 1 < \Re z < \alpha. \tag{3.19}$$

3.3 Variation Diminishing Transforms

Let $F \in \mathcal{E}$ with $F(0) \neq 0$, and let g be defined by $F^{-1}(z) = \mathcal{L}[g](z)$ in an open strip containing the origin. We will in addition assume that $g(t)$ is analytic on a set containing the real line.

As we will see in Section 3.5, the extremal functions $G_n(z)$ of $\operatorname{sgn}(x) z^n$ have the representation

$$G_n(x) - \operatorname{sgn}(x) x^n = -F(x) x^n \int_{-\infty}^{\infty} (\operatorname{sgn}(x) + \operatorname{sgn}(t)) e^{-xt} g_{(n-1)}(t) dt \tag{3.20}$$

in $x \neq 0$, where $g_{(m)}(t)$ is the tail

$$g_{(m)}(t) := g(t) - \sum_{k=0}^{m} \frac{g^{(k)}(0)}{k!} t^k.$$

The function $G_n(x)$ is a majorant or a minorant of $\operatorname{sgn}(x) x^n$ on the real line if the expression in (3.20) is of one sign for all x. In this section we shall prove that the tail $g_{(n-1)}(t)$ has at most $n+1$ real zeros. The condition $g^{(n)}(0) = 0$ will then imply that $g_{(n-1)}(t)$ has a zero of order $n+1$ at the origin, and no zero in $\mathbb{R}\backslash\{0\}$. An investigation of the sign changes of $x^n(\text{sgn}(x)+\text{sgn}(t))g_{(n-1)}(t)$ for real x and t shall show that the difference in (3.20) does not have a sign change on $\mathbb{R}$.

An important result of I. J. Schoenberg [19] states that the functions $g(t)$ considered in Theorem 3.2.4 have the property that for any bounded, continuos function $\varphi : \mathbb{R} \to \mathbb{R}$ the convolution $g * \varphi$ has no more changes of sign than φ has. A function with this property is called *variation diminishing*.

We shall show in this section that for any real polynomial P the sum $g + P$ has no more than $\deg(P) + 2$ real zeros. Since the tail $g_{(m)}(t)$ defined above is the sum of $g(t)$ and a polynomial of degree m, it follows that $g_{(m)}(t)$ has at most $m + 2$ real zeros.

Variation diminishing functions have been investigated in various places in the literature, among them articles by Schoenberg [19], [20], [21], and [4], and books by Hirschman and Widder [6] and by Karlin [9]. Here we give some results that are needed later. The presentation is based on [6], Chapter IV and on the article [19].

Definition 3.3.1. Denote by $\mathcal{V}[a(1), ..., a(n)]$ the number of changes of sign in the real sequence $a(1), ..., a(n)$.

Here zero values do not count as changes of sign in the sense that

$$\mathcal{V}[1, 0, 1] = 0$$

and

$$\mathcal{V}[1, 0, -1] = 1.$$

Definition 3.3.2. If $f(x)$ is a real function defined on $\mathbb{R}$ then

$$\mathcal{V}[f] := \sup\{\mathcal{V}[f(x_1), ..., f(x_n)] \mid -\infty < x_1 < ... < x_n < \infty,\ n \in \mathbb{N}\}.$$

It is possible that $\mathcal{V}[f] = \infty$.

Definition 3.3.3. A non-negative, integrable function $g : \mathbb{R} \to \mathbb{R}$ is said to be *variation diminishing* if $\mathcal{V}[\varphi * g] \leq \mathcal{V}[\varphi]$ for every bounded, continuous function $\varphi : \mathbb{R} \to \mathbb{R}$.

The following is the theorem of Schoenberg mentioned above (cf. [19] and [6], Chapter IV):

Theorem 3.3.4. *An integrable function $G : \mathbb{R} \to \mathbb{R}$ is variation diminishing if and only if $G(t) = g(t)$ almost everywhere, where g satisfies*

$$F^{-1}(z) = \mathcal{L}[g](z)$$

in an open strip containing the origin for some $F \in \mathcal{E}$ with $F(0) \neq 0$.

Proof. We shall prove only one direction of the theorem, namely that a function $g(t)$ satisfying $F^{-1}(z) = \mathcal{L}[g](z)$ in an open strip containing the origin for some $F \in \mathcal{E}$ with $F(0) \neq 0$ is variation diminishing. The converse implication of the theorem is not needed. The proof follows [6], Chapter IV.

Theorem 3.2.4 states that any such function $g(t)$ is the pointwise limit of repeated convolutions of functions of the form $|a|\mathfrak{g}(a(t+b))$ where a and b are real numbers and $a \neq 0$. (Recall that $\mathfrak{g}(t) = e^t$ for $t < 0$ and $\mathfrak{g}(t) = 0$ for $t > 0$.)

Therefore, we can establish the result by proving that $\mathfrak{g}(t)$ is variation diminishing, and that the set of variation diminishing functions is closed under repeated convolutions and pointwise limits.

Let us start by considering the example $F(z) := 1 - z$ in equation (3.3). Recall that

$$(1-z)^{-1} = \mathcal{L}[\mathfrak{g}](z) \text{ for } \Re z < 1,$$

where $\mathfrak{g}(t) = e^t$ for $t < 0$ and $\mathfrak{g}(t) = 0$ for $t > 0$. Let φ be a bounded, continuous, real-valued function on $\mathbb{R}$. We can verify directly that the convolution

$$f(x) := \mathfrak{g} * \varphi(x) = \int_x^\infty e^{x-t}\varphi(t)dt = e^x \int_x^\infty e^{-t}\varphi(t)dt \tag{3.21}$$

satisfies the differential equation

$$(1-D)f(x) = \varphi(x) \text{ for all } x \in \mathbb{R} \tag{3.22}$$

with the differential operator $D = d/dx$. Since φ is bounded, equation (3.21) implies $f(x) = \mathcal{O}(1)$ as $x \to \infty$.

Let f have $n > 0$ changes of sign on the real line, i.e. assume that there exist $x_0, ..., x_n \in \mathbb{R}$ with the property that f has alternating signs at any two consecutive values x_i. Since e^{-x} is positive on $\mathbb{R}$, the function $e^{-x}f(x)$ has the same property. Equation (3.22) can be written in the form

$$e^x D[e^{-x} f(x)] = \varphi(x). \tag{3.23}$$

The estimate $e^{-x}f(x) = o(1)$ as $x \to \infty$ together with Rolle's theorem implies that $D[e^{-x}f(x)]$ has at least n changes of sign, and so by (3.23), the function φ has at least n changes of sign on the real line.

This means that the convolution $\varphi * \mathfrak{g}$ has no more changes of sign than the function φ.

As shown above, the function $\mathfrak{g}$ is variation diminishing. It follows that the functions $T_a\mathfrak{g}(x) := |a|\mathfrak{g}(ax)$ are variation diminishing for every non-zero $a \in \mathbb{R}$. The following lemma ([6], Lemma IV.2.1a) shows that finite convolutions of functions $T_a\mathfrak{g}$ are variation diminishing as well.

Lemma 3.3.5. *The convolution of two variation diminishing functions is variation diminishing.*

Proof. This follows from the definition and the associativity of the convolution. □

Lemma 3.3.5 and equation (3.5) imply that the inverse Laplace transform of the reciprocal of any polynomial P having only real roots and satisfying $P(0) \neq 0$ is variation diminishing. The following lemma allows us to extend this statement to inverse Laplace transforms of reciprocals of functions $F \in \mathcal{E}$ with $F(0) \neq 0$ (cf. [6], Lemma IV.2.1b).

Lemma 3.3.6. *Let ψ, $(\psi_k)_{k=1}^{\infty}$ be real functions defined on $\mathbb{R}$, and let n be a non-negative integer. If the condition $\lim \psi_k(t) = \psi(t)$ holds in the pointwise sense, and*

$$\mathcal{V}[\psi_k] \leq n \textit{ for all } k \in \mathbb{N},$$

then

$$\mathcal{V}[\psi] \leq n.$$

Proof. If ψ has no changes of sign, the result is obvious. Assume ψ has $m > 0$ sign changes. Then there exists a sequence of $m + 1$ points for which ψ assumes alternating signs. By making k large enough the same will be true for ψ_k, so $\mathcal{V}[\psi_k] \geq m$. □

Let $F \in \mathcal{E}$ and let g be defined by $F^{-1}(z) = \mathcal{L}[g](z)$ in an open strip containing the origin. By Theorem 3.2.4 the function g is the pointwise limits of finite convolutions of the form (3.5). We obtain from Lemma 3.3.6 that g is variation diminishing. This finishes the direction of Theorem 3.3.4 that we wanted to prove. $\square$

The variation diminishing property is *a priori* applicable only to bounded, continuous functions φ. The next lemma extends this applicability to all continuous functions φ for which the convolution of φ with a variation diminishing function exists. The statement and proof of the lemma are from [6], Lemma IV.4.1.

Lemma 3.3.7. *If $g : \mathbb{R} \to \mathbb{R}$ is a variation diminishing function, $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, and $t \mapsto \varphi(t)g(x-t) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$, then*

$$\mathcal{V}[g * \varphi] \leq \mathcal{V}[\varphi].$$

Proof. Let $m \in \mathbb{N}_0$. We define the Fejér kernel $K(t)$ by

$$K(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.24}$$

Then by Lebesgue's Theorem of dominated convergence

$$\begin{aligned} g * \varphi(x) &= \lim_{T \to \infty} \int_{-\infty}^{\infty} K(t/T) g(x-t) \varphi(t) dt \\ &=: \lim_{T \to \infty} G_T(x). \end{aligned}$$

If $\mathcal{V}[\varphi] \leq m$, then $\mathcal{V}[K(t/T)\varphi(t)] \leq m$ for all $T > 0$ since $K \geq 0$. The function $G_T(t)$ is the convolution of $g(t)$ and $K(t/T)\varphi(t)$. Since g is variation diminishing and $K(t/T)\varphi(t)$ is bounded and continuous, it follows that $\mathcal{V}[G_T] \leq m$ for all $T > 0$. From Lemma 3.3.6 we obtain that $\mathcal{V}[g * \varphi] \leq m$. $\square$

Let $F \in \mathcal{E}$ and define g by $F^{-1}(z) = \mathcal{L}[g](z)$. By Lemma 3.2.9 the function g from Theorem 3.2.4 decreases exponentially fast as $|t| \to \infty$. In particular, we can apply the variation diminishing property to functions φ with a subexponential growth on the real line, i.e. $\varphi(t) = e^{o(|x|)}$.

This implies that all moments $\int_{\mathbb{R}} t^k g(t)dt$ exist. In this case the integral convolution $g * Q(x)$ is defined for every real polynomial Q and every $x \in \mathbb{R}$, and it represents a function P which is again a real polynomial, as we shall see in Lemma 3.3.8. In the remaining part of this section we investigate the relations between the two polynomials P and Q defined in this way.

Lemma 3.3.8. *Let $g : \mathbb{R} \to \mathbb{R}$ be such that all moments*

$$\mu_k := \int_{\mathbb{R}} t^k g(t)dt$$

for $k \in \mathbb{N}_0$ exist as Lebesgue integrals and such that $\mu_0 \neq 0$. Let P be a polynomial of degree $n \in \mathbb{N}_0$. There exists a polynomial Q of degree n such that $P = g * Q$. Conversely, for every polynomial Q of degree n there exists polynomial P of the same degree such that $P = g * Q$.

For fixed $n \in \mathbb{N}_0$ the coefficients of P are continuous functions of the coefficients of Q, and vice versa.

Proof. (This proof is part of the proof of Lemma 7 in [19].) Performing a Taylor expansion of P about x we obtain

$$\begin{aligned} Q * g(x) &= \int_{-\infty}^{\infty} g(t)Q(x-t)dt = \sum_{j=0}^{n} \frac{Q^{(j)}(x)}{j!}(-1)^j \int_{-\infty}^{\infty} g(t)t^j dt \\ &= \sum_{j=0}^{n} \frac{Q^{(j)}(x)(-1)^j \mu_j}{j!}. \end{aligned}$$

The term x^n occurs only for $j = 0$, so the condition $\mu_0 \neq 0$ implies that P and Q are polynomials of the same degree. If P and Q are given by

$$\begin{aligned} P(x) &= \sum_{k=0}^{n} a_k \binom{n}{k} x^k \\ Q(x) &= \sum_{k=0}^{n} b_k \binom{n}{k} x^k, \end{aligned}$$

then the kth coefficient of P has the representation

$$\binom{n}{k} a_k = \sum_{j=0}^{n-k} \frac{\mu_j(-1)^j}{j!} b_{k+j} \binom{n}{k+j} (k+j)(k+j-1)...(k+1),$$

and since $\binom{n}{k+j}\frac{(k+j)(k+j-1)...(k+1)}{j!} = \binom{n}{k}\binom{n-k}{j}$, the relation between P and Q can be written as the matrix multiplication

$$\begin{aligned} a_n &= \mu_0 b_n \\ a_{n-1} &= -\mu_1 b_n + \mu_0 b_{n-1} \\ a_{n-2} &= \mu_2 b_n - 2\mu_1 b_{n-1} + \mu_0 b_{n-2} \\ &\vdots \\ a_0 &= (-1)^n \mu_n b_n + ... + (-1)^{(j)}\mu_j \binom{n}{j} b_j + ... + \mu_0 b_0 \end{aligned} \tag{3.25}$$

and since $\mu_0 \neq 0$, this is a non-singular matrix transformation. Thus, for fixed n the sequences (a_k) and (b_k) are continuous functions of each other. □

We are now in a position to establish our estimate of the number of real zeros of $g + P$. Let $S \subseteq \mathbb{C}$, and let f be a function that is analytic on an open set $G \subseteq \mathbb{C}$ containing S. Define

$$Z_S[f] := \text{the number of zeros of } f \text{ in } S \text{ counted with multiplicity.} \tag{3.26}$$

Theorem 3.3.9. *Let $g(t)$ be the Laplace inverse of the reciprocal of some $F \in \mathcal{E}$ in an open strip S containing the origin, and assume that g is analytic in an open set containing the real line. Let $P(t)$ be a real polynomial which is not identically zero, and let $Q(t)$ be the polynomial determined by $P = g * Q$ as in Lemma 3.3.8. Then*

$$Z_{\mathbb{R}}[g + P] \leq Z_{\mathbb{R}}[Q] + 2. \tag{3.27}$$

In particular (since $\deg(P) = \deg(Q)$*), $Z_{\mathbb{R}}[g + P] \leq \deg(P) + 2$.*

Proof. Suppose first that the real zeros of $g + P$ are all simple. In this case the number of real zeros equals the number of sign changes of $g + P$. With aid of an approximate identity we shall express $g + P$ as a convolution of g with a function of which we can estimate the number of sign changes. Let $K(t)$ be the Fejér kernel defined in (3.24). Set $K_\varepsilon(t) = 1/\varepsilon \cdot K(t/\varepsilon)$ and recall that K_ε is an approximate identity for $\varepsilon \to 0+$. We have

$$g + P = \lim_{\varepsilon \to 0+} g * K_\varepsilon + g * Q = \lim_{\varepsilon \to 0+} g * (K_\varepsilon + Q). \tag{3.28}$$

The graph of K_ε forms a triangle with base length 2ε and height $1/\varepsilon$.

So for sufficiently small ε the graph of the polynomial $-Q$ intersects this triangle at most twice. This means that $K_\varepsilon + Q$ has at most $\mathcal{V}[Q] + 2$ sign changes, namely those sign changes of Q which are not at the origin and the two possible intersections. By Theorem 3.3.4 the function g is variation diminishing. Thus, with equation (3.28) and Lemma 3.3.6 for sufficiently small $\varepsilon > 0$, we obtain

$$Z_{\mathbb{R}}[g + P] \le \mathcal{V}[g + P] \le \mathcal{V}[K_\varepsilon + Q] \le Z_{\mathbb{R}}[Q] + 2, \qquad (3.29)$$

which is the desired conclusion in this case.

Next, suppose that the real zeros of $g + P$ are not all simple. Here the number of sign changes is smaller than the number of real zeros, and we cannot apply the variation diminishing property directly. Instead we proceed by constructing a perturbation $g+P+P_1$ of $g+P$ with as many sign changes as $g + P$ has real zeros. Let Q_1 be defined by $P_1 = g * Q_1$. We construct P_1 in such a way that $Q + Q_1$ has no more sign changes than the polynomial Q has real zeros. An application of (3.29) with $P + P_1$ and $Q + Q_1$ instead of P and Q will finish the proof of the theorem.

Lemma 3.3.10. *Let g be as in Theorem 3.3.9. Let P and Q be non-zero, real polynomials satisfying $P = g * Q$. Then there exist real polynomials P_1 and Q_1 with $P_1 = g * Q_1$ such that the inequalities $Z_{\mathbb{R}}[g + P] \le \mathcal{V}[g + P + P_1]$ and $\mathcal{V}[Q + Q_1] \le Z_{\mathbb{R}}[Q]$ hold.*

Before we give the proof of this lemma we shall finish the proof of Theorem 3.3.9. We have

$$g + P + P_1 = \lim_{\varepsilon \to 0+} g * K_\varepsilon + g * (Q + Q_1) = \lim_{\varepsilon \to 0+} g * (K_\varepsilon + Q + Q_1),$$

and therefore

$$\mathcal{V}[g + P + P_1] \le \mathcal{V}[Q + Q_1] + 2.$$

By Lemma 3.3.10

$$Z_{\mathbb{R}}[g + P] \le \mathcal{V}[g + P + P_1] \le \mathcal{V}[Q + Q_1] + 2 \le Z_{\mathbb{R}}[Q] + 2,$$

which concludes the proof of Theorem 3.3.9. ☐

Proof of Lemma 3.3.10. We construct the polynomial P_1 by changing $g + P$

to $g+P+E$ with an error term E in such a fashion that $g+P+E$ has only simple real zeros. We approximate E by a polynomial P_1, so that Lemma 3.3.8 is applicable, yielding a polynomial Q_1 with $P_1 = g * Q_1$. Finally we investigate the number of real zeros of $Q + Q_1$.

Since the Laplace transform of $g(t)$ converges absolutely in a neighborhood of the origin, the function $g(t)$ decreases exponentially as $t \to \pm\infty$. Since $g(t)$ is analytic in a region containing the real line and P is a non-zero polynomial, the function $g+P$ has only finitely many real zeros. Let $(z_i)_{i=0}^n$ be the list of these zeros such that every zero is repeated according to its multiplicity. We can represent $g+P$ in the form

$$g(t) + P(t) = (t - z_0)(t - z_1)...(t - z_n)h(t), \tag{3.30}$$

where $h(t)$ is a function which is non-zero on $\mathbb{R}$ and analytic on a set containing $\mathbb{R}$. For real numbers ε_i we define a perturbation of $g+P$ by

$$p_\varepsilon(t) := (t - z_0 - \varepsilon_0)...(t - z_n - \varepsilon_n)h(t). \tag{3.31}$$

By choosing $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$ and letting $\varepsilon_i \in (0, a)$ for some sufficiently small $a > 0$ we can assume that p_ε has exactly $n+1$ simple zeros (i.e. $n+1$ sign changes) $z_0 + \varepsilon_0, ..., z_n + \varepsilon_n$ on the real line. In order to keep the notation simple we set $\varepsilon_i := \varepsilon\eta_i$ with fixed values $\eta_i \in (0,1)$ and a parameter ε in $(0, a)$. We have

$$(t - z_0 - \varepsilon\eta_0)...(t - z_n - \varepsilon\eta_n) = \prod_i (t - z_i) + \varepsilon L(t, \varepsilon),$$

where $L(t, \varepsilon)$ is a polynomial in t and in ε.

Set $E_\varepsilon(t) := L(t, \varepsilon)h(t)$, where $h(t)$ is defined in (3.30). We obtain from (3.31)

$$p_\varepsilon(t) = g(t) + P(t) + \varepsilon E_\varepsilon(t).$$

We construct a polynomial H_ε which interpolates $E_\varepsilon(t)$ and $E'_\varepsilon(t)$ at the points $t = z_i + \varepsilon_i$ and which is of the same sign for large $|t|$ as $P(t)$ is.

Let $H_{1,\varepsilon}(t)$ be a polynomial which interpolates $E_\varepsilon(t)$ and $E'_\varepsilon(t)$ at the points $t = z_i + \varepsilon_i$ in such a way that the coefficients of $H_{1,\varepsilon}$ are continuous functions of ε. Let $m > \deg(H_{1,\varepsilon})$ such that $m \equiv \deg(P)$ modulo 2. Let

$\sigma \in \mathbb{R}\backslash\{0\}$ be the leading coefficient of P and define a polynomial H_ε by

$$H_\varepsilon(t) := H_{1,\varepsilon}(t) + \sigma t^m \prod_{k=0}^{n} (t - z_k - \varepsilon\eta_k)^2.$$

By construction, the polynomial H_ε interpolates $E_\varepsilon(t)$ and $E'_\varepsilon(t)$ at the points $t = z_i + \varepsilon\eta_i$, since the values $H_\varepsilon(z_i + \varepsilon\eta_i)$ are determined by $H_{1,\varepsilon}$ alone. Therefore, $g(t) + P(t) + \varepsilon H_\varepsilon$ has a simple zero at any point $t = z_i + \varepsilon_i$. Since sign changes occur at these zeros, we obtain

$$Z_{\mathbb{R}}[g + P] \leq \mathcal{V}[g + P + \varepsilon H_\varepsilon] \tag{3.32}$$

for all $\varepsilon \in (0, a)$.

Define Q_ε by $H_\varepsilon = g * Q_\varepsilon$. By construction, $H_\varepsilon(t)$ has leading coefficient $\sigma \neq 0$ for all $\varepsilon \in (0, a)$. By Lemma 3.3.8 the leading coefficient of Q_ε is therefore non-zero and independent of ε. Thus, for all $\varepsilon \in [0, a)$ the polynomial Q_ε is not identically zero. By Hurwitz's theorem there is a $K > 0$ such that for sufficiently small $\varepsilon > 0$ all complex zeros of Q_ε are inside the disc $|z| \leq K$, and we can assume that the zeros of Q are also in this disc. Since $Q + \varepsilon Q_\varepsilon \to Q \not\equiv 0$ as $\varepsilon \to 0$, Hurwitz's theorem applied to a set S containing $[-K, K]$ and all real but no non-real zeros of Q implies that

$$Z_{[-K,K]}[Q + \varepsilon Q_\varepsilon] \leq Z_{[-K,K]}[Q]. \tag{3.33}$$

By the construction of H_ε, the degree of P and the degree of H_ε have the same parity, and the leading coefficient of P equals the leading coefficient of H_ε. Lemma 3.3.8 implies that the degree of Q_ε and the degree of Q have the same parity, and that the leading coefficients of Q and Q_ε are equal. Since for sufficiently small $\varepsilon > 0$ the polynomials Q_ε and Q have no real zeros on $|t| \geq K$, we obtain the inequality $|Q + Q_\varepsilon| > |Q| > 0$ in $|t| \geq K$, i.e. $Q + \varepsilon Q_\varepsilon$ has no real zeros in the region $|t| \geq K$. From (3.33) we obtain

$$Z_{\mathbb{R}}[Q + \varepsilon Q_\varepsilon] \leq Z_{\mathbb{R}}[Q]. \tag{3.34}$$

We set $P_1 = \varepsilon H_\varepsilon$ and $Q_1 = \varepsilon Q_\varepsilon$ for sufficiently small ε. Equations (3.32) and (3.34) complete the proof. □

As a first corollary of Theorem 3.3.9 we obtain a bound on the number of real zeros of the tail $g_{(n)}$, where $n \in \mathbb{N}_0$.

Corollary 3.3.11. *Let $m \in \mathbb{N}_0$, let g be as in Theorem 3.3.9, and assume $g^{(m)}(0) \neq 0$. Then*

$$g_{(m)}(t) = g(t) - \sum_{k=0}^{m} \frac{g^{(k)}(0)}{k!} t^k$$

has $m+2$ real zeros. Depending on the sign of $g^{(m)}(0)g^{(m+1)}(0)$ there are three cases:

1. *If $g^{(m+1)}(0) = 0$ then $g_{(m)}(t)$ has a zero of order $m+2$ at $t = 0$ and no zeros in $\mathbb{R}\backslash\{0\}$.*
2. *If $g^{(m)}(0)g^{(m+1)}(0) > 0$ then $g_{(m)}(t)$ has a zero of order $m+1$ at the origin, precisely one simple zero in $(0, \infty)$, and no zeros in $(-\infty, 0)$.*
3. *If $g^{(m)}(0)g^{(m+1)}(0) < 0$ then $g_{(m)}(t)$ has a zero of order $m+1$ at the origin, precisely one simple zero in $(-\infty, 0)$, and no zeros in $(0, \infty)$.*

Proof. Let

$$P(t) = -\sum_{k=0}^{m} \frac{g^{(k)}(0)}{k!} t^k.$$

Since $g_{(m)} = g + P$, Theorem 3.3.9 implies $Z_{\mathbb{R}}[g_{(m)}] \leq m+2$. If $g^{(m+1)}(0) = 0$ then $g_{(m)}(t)$ has already a zero of order $m+2$ at the origin, thus $g_{(m)}(t)$ cannot have another zero in $\mathbb{R}\backslash\{0\}$.

If $g^{(m+1)}(0) \neq 0$, then $g_{(m)}$ has a zero of order $m+1$ at the origin, and therefore exactly one zero in $\mathbb{R}\backslash\{0\}$ which has to be simple. Moreover, $g_{(m)}(t) \sim -g^{(m)}(0)t^m/m!$ as $|t| \to \infty$, because g is converging to zero as $t \to \pm\infty$ by Lemma 3.2.9.

Therefore, the behavior of $g_{(m)}(t)$ at the origin is determined by the sign of $g^{(m+1)}(0)$, and the behavior of $g_{(m)}(t)$ as $t \to \pm\infty$ is determined by the sign of $-g^{(m)}(0)$.

If $g^{(m+1)}(0) > 0$, then $g(t) > 0$ for $t > 0$ near the origin. If $g^{(m)}(0) > 0$, then $g(t) < 0$ for large t. Therefore $g(t)$ has a sign change in $(0, \infty)$. Since $g(t)$ has exactly one (simple) zero in $\mathbb{R}\backslash\{0\}$, the function $g(t)$ has no other zeros in $\mathbb{R}\backslash\{0\}$.

The other cases are handled similarly. □

The following corollary can be found in [6] IV.5. The proof given here differs slightly from the proof given in [6].

Corollary 3.3.12. *Let g be as in Theorem 3.3.9, and let $n \in \mathbb{N}_0$. Then $g^{(n)}(t)$ has exactly n simple real zeros and no multiple real zeros.*

Proof. Let $\varepsilon > 0$ and define $E_\varepsilon(t) := \pi^{-1/2}\varepsilon^{-1}\exp(-(t/\varepsilon)^2)$ where $t \in \mathbb{R}$. Since E_ε is an approximate identity for $\varepsilon \to 0+$, we have for $n \in \mathbb{N}_0$ the representation

$$g^{(n)} = \lim_{\varepsilon \to 0} g * E_\varepsilon^{(n)}.$$

By [24], Theorem 6.3, the function $E_\varepsilon^{(n)}$ has exactly n changes of sign. Since g is variation diminishing, the function $g^{(n)}$ has no more than n changes of sign by Lemma 3.3.6.

Recall that

$$\lim_{t \to -\infty} g^{(n)}(t) = \lim_{t \to \infty} g^{(n)}(t) = 0 \tag{3.35}$$

for all $n \in \mathbb{N}_0$ by Lemma 3.2.9

By Remark 3.2.6 the function $g(t)$ has no odd order zeros. If g has an even order zero, then Rolle's theorem and (3.35) imply that $g'(t)$ has at least two changes of sign. However, by the first part of the proof $g'(t)$ has at most one change of sign. Therefore $g(t)$ has no zeros on the real line.

Assume now that $g^{(n)}(y) = g^{(n+1)}(y) = 0$ for some $n \in \mathbb{N}$ and $y \in \mathbb{R}$. By Taylor's theorem

$$g(t) = \sum_{i=0}^{n-1} \frac{g^{(\ell)}(y)}{\ell!}(t-y)^\ell + (t-y)^{n+2}h(t),$$

where $h(t)$ is a function which is analytic on a set containing the real line.

Let $P(t) = \sum_{i=0}^{n-1} \frac{g^{(\ell)}(y)}{\ell!}(t-y)^\ell$. Since $g(y) \neq 0$, P is a non-zero polynomial. It satisfies

$$g(x) - P(x) = (x-y)^{n+2}h(x),$$

and its degree is at most $n-1$. Therefore, $g - P$ has at least $\deg(P) + 3$ real zeros. This is a contradiction to Theorem 3.3.9. Thus, for $n \geq 1$ all real zeros of $g^{(n)}(t)$ are simple.

On the other hand, from (3.35) and Rolle's theorem we obtain that $g'(t)$ has at least one zero. It follows inductively that $g^{(n)}(t)$ has at least n real zeros. Combining these facts we obtain that $g^{(n)}(t)$ has n simple real zeros and no other real zeros. $\square$

Remark 3.3.13. If g is analytic on a set containing the real line, and its Laplace transform represents the reciprocal of $F \in \mathcal{E}$ on a strip $\mathfrak{a} < \Re z < \mathfrak{b}$ (which does not necessarily include the origin), then g has no real zeros. This follows from the fact that the function $F^{-1}(z-\gamma)$ for $\gamma \in (\mathfrak{a}, \mathfrak{b})$ is represented by the Laplace transform of a function g_1 in a strip including the origin. By the previous lemma g_1 has no real zeros. Since $g(t) = e^{\gamma t} g_1(t)$, the claim follows.

Given $F \in \mathcal{E}$ and two consecutive elements $\mathfrak{a} < \mathfrak{b}$ in S_F (defined in (3.10)), we obtain as a consequence of Lemma 3.2.9 that the function g defined by $F^{-1}(z) = \mathcal{L}[g](z)$ in $\mathfrak{a} < \Re z < \mathfrak{b}$ is integrable if $0 \in (\mathfrak{a}, \mathfrak{b})$. In the next lemma we shall show that the condition $0 \notin (\mathfrak{a}, \mathfrak{b})$ implies that g is a strictly monotone function.

Lemma 3.3.14. *Let $F \in \mathcal{E}$, and assume that F is not of the form $e^{bz}P(z)$ with a polynomial $P(z)$. Let $\mathfrak{a} < \mathfrak{b}$ be two consecutive elements in S_F. By Theorem 3.2.4 there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying $F^{-1}(z) = \mathcal{L}[g](z)$ in $\mathfrak{a} < \Re z < \mathfrak{b}$. If $0 \notin (\mathfrak{a}, \mathfrak{b})$, then $g(t)$ is a monotone function. If in addition g is analytic on a set containing the real line, then $g'(t) \neq 0$ for all real t.*

Proof. Assume first that $\mathfrak{a} \geq 0$, and define $F_{\mathfrak{a}}(z) := F(z + \mathfrak{a})$. We have

$$F_{\mathfrak{a}}^{-1}(z) = \int_{-\infty}^{\infty} g(t) e^{-\mathfrak{a}t} e^{-zt} dt \text{ in } 0 < \Re z < \mathfrak{b} - \mathfrak{a}.$$

Since F has at least three zeros, $e^{-\mathfrak{a}t} g(t)$ has a continuous derivative by Lemma 3.2.9. The function $z^{-1}F_{\mathfrak{a}}(z)$ is an element of $\mathcal{E}$. Since

$$zF_{\mathfrak{a}}^{-1}(z) = \int_{-\infty}^{\infty} e^{-zt} \frac{d}{dt}[e^{-\mathfrak{a}t} g(t)] dt \text{ in } 0 < \Re z < \mathfrak{b} - \mathfrak{a}$$

we obtain from Remark 3.2.6 that $(d/dt)[e^{-\mathfrak{a}t}g(t)]$ does not have a change of sign on the real line. We claim that

$$g'(t) = \frac{d}{dt}[e^{\mathfrak{a}t} e^{-\mathfrak{a}t} g(t)] = e^{\mathfrak{a}t} \Big(\frac{d}{dt}[e^{-\mathfrak{a}t} g(t)] + \mathfrak{a} e^{-\mathfrak{a}t} g(t) \Big) \tag{3.36}$$

is of one sign, and that this expression is non-zero if g is analytic on a set containing the real line.

By Remark 3.2.6 the sign of $(d/dt)[e^{-\mathfrak{a}t}g(t)]$ equals the sign of $(x-\mathfrak{a})^{-1}F(x)$ in $\mathfrak{a} < x < \mathfrak{b}$, which is equal to the sign of $F(x)$ on that interval. The function $e^{-\mathfrak{a}t}g(t)$ does not have a change of sign either, and since the sign of $e^{-\mathfrak{a}t}g(t)$ is that of $F(x)$ in $\mathfrak{a} < x < \mathfrak{b}$ as well, we obtain from 3.36 that $g'(t)$ does not have a change of sign.

If in addition g is analytic on a set containing the real line, then by Remark 3.3.13 we have $e^{-\mathfrak{a}t}g(t) > 0$. In this case we have strict inequality in (3.36) for all real t.

If $\mathfrak{b} \leq 0$, we consider the function $F^*(z) := F(-z)$. Since

$$F^*(-z) = \int_{-\infty}^{\infty} g(t)e^{zt}dt = \int_{-\infty}^{\infty} g(-t)e^{-zt}dt,$$

we obtain from the first case that $g(-t)$ is monotone (and strictly monotone if $g(-t)$ is analytic), thus the claim of the lemma follows in this case as well. ☐

Remark 3.3.15. By Lemma 3.2.9, g is integrable if $0 \in (\mathfrak{a}, \mathfrak{b})$. Therefore, Lemma 3.3.14 implies that g is either integrable or monotone. If g is analytic on a set containing the real line, g is integrable or $g' > 0$.

3.4 The Laplace Transform

The following lemmata collect facts about the Laplace transform that are needed in Section 3.5.

Suppose $f \in C^\infty(\mathbb{R})$ satisfies for each $n \in \mathbb{N}_0$

$$f^{(n)}(t) \ll |P_n(t)|e^{At} \text{ as } t \to \infty, \tag{3.37}$$

$$f^{(n)}(t) \ll |Q_n(t)|e^{Bt} \text{ as } t \to -\infty, \tag{3.38}$$

where P_n and Q_n are polynomials.

Lemma 3.4.1. *Consider the equations*

$$\int_0^\infty e^{-zt}f(t)dt = \sum_{j=0}^m \frac{f^{(j)}(0)}{z^{j+1}} + \frac{1}{z^{m+1}} \int_0^\infty e^{-zt}f^{(m+1)}(t)dt, \tag{3.39}$$

$$-\int_{-\infty}^0 e^{-zt}f(t)dt = \sum_{j=0}^m \frac{f^{(j)}(0)}{z^{j+1}} - \frac{1}{z^{m+1}} \int_{-\infty}^0 e^{-zt}f^{(m+1)}(t)dt. \tag{3.40}$$

If $f \in C^\infty(\mathbb{R})$ satisfies (3.37), *then* (3.39) *holds in the region $\Re z > A$ for every $m \in \mathbb{N}_0$.*

If $f \in C^\infty(\mathbb{R})$ satisfies (3.38), *then* (3.40) *holds in the region $\Re z < B$ for every $m \in \mathbb{N}_0$.*

Proof. The conditions (3.37) and (3.38) imply the absolute convergence of these integrals in the indicated regions.

Let $f \in C^\infty(\mathbb{R})$ satisfy (3.37), and let $\Re z > A$. We have

$$\begin{aligned}\int_0^\infty e^{-zt} f(t)dt &= -\frac{1}{z} e^{-zt} f(t)\Big|_0^\infty + \frac{1}{z}\int_0^\infty e^{-zt} f'(t)dt \\ &= \frac{f(0)}{z} + \frac{1}{z}\int_0^\infty e^{-zt} f'(t)dt,\end{aligned}$$

since the integrals converge absolutely. Repeated application of this identity proves the first part of the lemma, and a similar computation shows the corresponding identity for $\Re z < B$. □

For real numbers x and t we define

$$\operatorname{sgn}(x,t) := \operatorname{sgn}(x) + \operatorname{sgn}(t) = \begin{cases} 2 & \text{if } x > 0 \text{ and } t > 0, \\ -2 & \text{if } x < 0 \text{ and } t < 0, \\ 0 & \text{if } xt < 0. \end{cases} \tag{3.41}$$

Lemma 3.4.2. *Let $z = x + iy \in \mathbb{C}$ with real numbers x and y. We have*

$$\begin{aligned}\sum_{k=0}^m a_k z^{m-k} &= z^{m+1} \int_0^\infty \sum_{k=0}^m \frac{a_k}{k!} t^k e^{-zt} dt \text{ in } \Re z > 0 \\ &= -z^{m+1} \int_{-\infty}^0 \sum_{k=0}^m \frac{a_k}{k!} t^k e^{-zt} dt \text{ in } \Re z < 0 \\ &= \frac{1}{2} z^{m+1} \int_{-\infty}^\infty \operatorname{sgn}(x,t) \sum_{k=0}^m \frac{a_k}{k!} t^k e^{-zt} dt \text{ in } \Re z \neq 0.\end{aligned}$$

Proof. Set $A = B = 0$ and

$$f(t) := \sum_{k=0}^m \frac{a_k}{k!} t^k$$

in Lemma 3.4.1, note that $a_k = f^{(k)}(0)$ and $f^{(m+1)}(t) = 0$ for all $t \in \mathbb{R}$, multiply the two identities in Lemma 3.4.1 by z^{m+1}, and combine them using (3.41). $\square$

Lemma 3.4.3. *Let $f \in C^\infty(\mathbb{R})$ satisfy* (3.37) *and* (3.38) *for $A = B = 0$, and define*

$$f_{(m)}(t) := f(t) - \sum_{j=0}^{m} \frac{f^{(j)}(0)}{j!} t^j.$$

We have for $\Re z \neq 0$

$$z^{m+1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) e^{-zt} f(t) dt - 2 \sum_{j=0}^{m} f^{(j)}(0) z^{m-j} \tag{3.42}$$

$$= \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) e^{-zt} f^{(m+1)}(t) dt \tag{3.43}$$

$$= z^{m+1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) e^{-zt} f_(m)(t) dt. \tag{3.44}$$

Proof. The equality of (3.42) and (3.43) is a combination of the two identities in Lemma 3.4.1 using (3.41). The equality of (3.42) and (3.44) follows after an application of Lemma 3.4.2 to the sum in (3.42). $\square$

Lemma 3.4.4. *Let $F \in \mathcal{E}$ such that $F^{-1}(z) = \mathcal{L}[g](z)$ in $\mathfrak{a} < \Re z < \mathfrak{b}$, where $\mathfrak{a} < \mathfrak{b}$ are two consecutive elements in S_F (cf.* (3.10)*), and let P be a polynomial with real coefficients. Then*

$$P(z) = \frac{F(z)}{z} \int_{-\infty}^{\infty} P(D) g'(t) e^{-zt} dt \text{ in the strip } \mathfrak{a} < \Re z < \mathfrak{b},$$

where $D = d/dt$ denotes the differential operator. Note that, if $0 \in (\mathfrak{a}, \mathfrak{b})$ then the integral on the right hand side vanishes at the origin.

Proof. By Lemma 3.2.9 we have

$$g^{(n)}(t) \ll |P_n(t)| e^{\mathfrak{a}t} \text{ as } t \to \infty \tag{3.45}$$

$$g^{(n)}(t) \ll |Q_n(t)| e^{\mathfrak{b}t} \text{ as } t \to -\infty, \tag{3.46}$$

with polynomials P_n and Q_n for all $n \in \mathbb{N}_0$. If $z \in \mathbb{C}$ satisfies $\Re z \in (\mathfrak{a}, \mathfrak{b})$

with $\Re z \neq 0$, then

$$\frac{1}{F(z)} = \int_{-\infty}^{\infty} g(t)e^{-zt}dt = g(t)\frac{e^{-zt}}{-z}\bigg|_{-\infty}^{\infty} - \frac{1}{-z}\int_{-\infty}^{\infty} g'(t)e^{-zt}dt$$
$$= \frac{1}{z}\int_{-\infty}^{\infty} g'(t)e^{-zt}dt.$$

We obtain

$$1 = \frac{F(z)}{z}\int_{-\infty}^{\infty} g'(t)e^{-zt}dt \text{ in } \mathfrak{a} < \Re z < \mathfrak{b} \tag{3.47}$$

If $0 \in (\mathfrak{a}, \mathfrak{b})$, the right-hand side is defined by continuity. Inductively, we see that

$$z^n = \frac{F(z)}{z}\int_{-\infty}^{\infty} g^{(n+1)}(t)e^{-zt}dt \text{ where } n \in \mathbb{N}_0. \tag{3.48}$$

Let $P(z) := \sum_{k=0}^{n} a_k z^k$ be a real polynomial. Equation (3.48) implies

$$\sum_{k=0}^{n} a_k z^k = \frac{F(z)}{z}\int_{-\infty}^{\infty} P(D)g'(t)e^{-zt}dt.$$

□

3.5 Entire Approximations

Let $g : \mathbb{R} \to \mathbb{R}$ be measurable. Recall the formal notation

$$\mathcal{L}[g](z) = \int_{-\infty}^{\infty} g(t)e^{-zt}dt.$$

Definition 3.5.1. We assume that $F \in \mathcal{E}$ satisfies $F^{-1}(z) = \mathcal{L}[g](z)$ for $g : \mathbb{R} \to \mathbb{R}$ in some strip $\mathfrak{a} < \Re z < \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are two consecutive elements in $S_F := \{b \in \mathbb{R} : F(b) = 0\} \cup \{\pm\infty\}$. In addition we assume that the function g is **analytic on a set containing the real line**. Let P be a polynomial with real coefficients. For $\mathfrak{a} < \Re z < \mathfrak{b}$ we define

$$G_{F,P}(z) := -\frac{F(z)}{z}\mathcal{L}[\text{sgn} \cdot P(D)g'](z). \tag{3.49}$$

If we are using one fixed function $F \in \mathcal{E}$, we suppress the subscript F and write G_P instead of $G_{F,P}$. If $P(z) = z^n$ for some $n \in \mathbb{N}_0$, we write $G_n(z)$ instead of $G_{z^n}(z)$.

It can be shown that the condition that g is analytic implies that $F(z)$ is not of the form $e^{bz}P(z)$ for a polynomial $P(z)$.

Theorem 3.5.2. *Let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1. If $F(0) = 0$, or if $P(D)g(0) = 0$, then $G_P(z)$ has an analytic continuation to the entire complex plane, and the estimate*

$$|G_P(x) - \operatorname{sgn}(x)P(x)| \ll |x|^{-2}|F(x)| \tag{3.50}$$

holds for $|x| > \max\{|\mathfrak{a}|, |\mathfrak{b}|\} + \delta$ with some $\delta > 0$.

Proof. If $P(z)$ is a polynomial with real coefficients given by

$$P(z) := \sum_{k=0}^{n} a_k z^k,$$

then by Lemma 3.4.4 we have the representations

$$\begin{aligned} G_P(z) &= P(z) - \frac{F(z)}{z}\int_{-\infty}^{\infty}(\operatorname{sgn}(t)+1)P(D)g'(t)e^{-zt}dt \text{ for } \mathfrak{a} < \Re z < \mathfrak{b} \\ &= P(z) - 2\frac{F(z)}{z}\int_{0}^{\infty} P(D)g'(t)e^{-zt}dt \qquad (3.51) \\ &= P(z) - 2F(z)\int_{0}^{\infty} P(D)g(t)e^{-zt}dt + 2\frac{F(z)}{z}P(D)g(0). \qquad (3.52) \end{aligned}$$

For the last equality we used Lemma 3.4.1. By Lemma 3.2.9 we have $P(D)g'(t) \ll e^{(\mathfrak{a}+\varepsilon)t}$ for every $\varepsilon > 0$ as $t \to \infty$, so (3.51) and (3.52) are valid in the region $\Re z > \mathfrak{a}$. Similarly,

$$\begin{aligned} G_P(z) &= -P(z) - \frac{F(z)}{z}\int_{-\infty}^{\infty}(\operatorname{sgn}(t)-1)P(D)g'(t)e^{-zt}dt \text{ for } \mathfrak{a} < \Re z < \mathfrak{b} \\ &= -P(z) + 2\frac{F(z)}{z}\int_{-\infty}^{0} P(D)g'(t)e^{-zt}dt \qquad (3.53) \\ &= -P(z) + 2F(z)\int_{-\infty}^{0} P(D)g(t)e^{-zt}dt + 2\frac{F(z)}{z}P(D)g(0), \qquad (3.54) \end{aligned}$$

and since $P(D)g'(t) \ll e^{(\mathfrak{b}-\varepsilon)t}$ for every $\varepsilon > 0$ as $t \to -\infty$, equations (3.53) and (3.54) are valid in the region $\Re z < \mathfrak{b}$.

By assumption, $F(0) = 0$ or $P(D)g(0) = 0$. Thus the representations (3.52) and (3.54) are analytic in their respective domains even if the origin lies in a domain. Since these halfplanes have the non-empty intersection $\mathfrak{a} < \Re z < \mathfrak{b}$, the function G_P has a continuation as an analytic function (which we also denote by G_P).

It remains to show (3.50). By Lemma 3.2.9, $P(D)g'(t)$ satisfies (3.37) and (3.38). Thus

$$\int_0^\infty P(D)g'(t)e^{-zt}dt \ll \int_0^\infty |P_n(t)|e^{(\mathfrak{a}-z)t}dt \ll |z|^{-1} \text{ for } \Re z \geq a > \mathfrak{a},$$
$$-\int_{-\infty}^0 P(D)g'(t)e^{-zt}dt \ll -\int_{-\infty}^0 |Q_n(t)|e^{(\mathfrak{b}-z)t}dt \ll |z|^{-1} \text{ for } \Re z \geq b > \mathfrak{b}.$$

Inserting these estimates in (3.51) and (3.53) proves (3.50). □

Let $z = x + iy$ with $x, y \in \mathbb{R}$. Recall $\operatorname{sgn}(x,t) := \operatorname{sgn}(x) + \operatorname{sgn}(t)$ for $x, t \in \mathbb{R}$ and note that $\operatorname{sgn}(x,t) = 0$ if $-xt > 0$. If $0 \in [\mathfrak{a}, \mathfrak{b}]$ in Theorem 3.5.2, then in particular equation (3.51) is valid in $\Re z > 0$, and (3.53) is valid in $\Re z < 0$. Thus we obtain

Lemma 3.5.3. *Let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1. If $0 \in [\mathfrak{a}, \mathfrak{b}]$, then*

$$G_P(z) - \operatorname{sgn}(x)P(z) = -\frac{F(z)}{z}\int_{-\infty}^{\infty} \operatorname{sgn}(x,t)P(D)g'(t)e^{-zt}dt \tag{3.55}$$

for all $z \in \mathbb{C}\backslash\{z : x = 0\}$.

Theorem 3.5.2 provides us with entire approximations $G_{F,P}$ to $\operatorname{sgn}(x)P(x)$ on the real line for a real polynomial P. Since we are interested in constructing majorants and minorants of $\operatorname{sgn}(x)P(x)$, we shall investigate conditions under which $G_{F,P}(x) - \operatorname{sgn}(x)P(x)$ has no changes of sign on the real axis.

We start by considering $P(z) = z^n$. We shall show that $\mathcal{L}[\operatorname{sgn} \cdot g^{(n+1)}](z)$ does not have changes of sign on the real axis, provided that either $F(0) = 0$ or $g^{(n)}(0) = 0$ (cf. Theorem 3.5.2). Recall that the tail of g is defined by

$$g_{(n)}(t) := g(t) - \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!}t^k. \tag{3.56}$$

By Lemma 3.4.3

$$-\frac{F(z)}{z}\int_{-\infty}^{\infty}\operatorname{sgn}(x,t)g^{(n+1)}(t)e^{-zt}dt = -F(z)x^n\int_{-\infty}^{\infty}\operatorname{sgn}(x,t)g_{(n)}(t)e^{-zt}dt.$$

Before we give the conditions under which $G_P(z)$ is a majorant of $\operatorname{sgn}(x)z^n$, we prove an auxiliary lemma about the sign of $g_{(n-1)}(t)$.

Lemma 3.5.4. *Let $n \in \mathbb{N}$, and let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1. Assume additionally that $0 \in (\mathfrak{a}, \mathfrak{b})$. If $g^{(n)}(0) = 0$, then*

$$g^{(n+1)}(0) \cdot x^n \operatorname{sgn}(x,t)g_{(n-1)}(t) \geq 0 \tag{3.57}$$

for all real x and t.

Proof. The function $\operatorname{sgn}(x,t) = \operatorname{sgn}(x) + \operatorname{sgn}(t)$ is positive if x and t are both positive, negative if x and t are both negative, and zero if $xt < 0$. We see inductively that for $n \in \mathbb{N}_0$

$$x^n t^{n+1}\operatorname{sgn}(x,t) \geq 0 \tag{3.58}$$

for all real x and t. Since $0 \in (\mathfrak{a}, \mathfrak{b})$, by Theorem 3.3.4 the function $g(t)$ is variation diminishing. By Corollary 3.3.12, the condition $g^{(n)}(0) = 0$ implies that $g^{(n-1)}(0) \neq 0$. Thus, Corollary 3.3.11 is applicable with $m = n-1$. We find that the function $g_{(n-1)}(t)$ has a zero of order $n+1$ at $t = 0$ and no other zero in $\mathbb{R}$, i.e. $\operatorname{sgn}\{g^{(n+1)}(0)g_{(n-1)}(t)\} = \operatorname{sgn}(t^{n+1})$ for all real t. It follows from (3.58) that

$$g^{(n+1)}(0) \cdot x^n \operatorname{sgn}(x,t)g_{(n-1)}(t) \geq 0 \tag{3.59}$$

for all real x and t. Note that $g^{(n+1)}(0)$ cannot equal zero, since by assumption $g^{(n)}(0) = 0$ holds, and every derivative of $g(t)$ has only simple zeros by another application of Corollary 3.3.12. ☐

Let $P(z) = z^n$ in equation Lemma 3.5.3. We are now in a position to determine the sign of the integral on the real line.

Lemma 3.5.5. *Let $n \in \mathbb{N}$, let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1 with $0 \in [\mathfrak{a}, \mathfrak{b}]$, and assume $g^{(n)}(0) = 0$. If $0 \in \{\mathfrak{a}, \mathfrak{b}\}$ assume in addition that*

$F(z)$ has a zero of order 2 at the origin.

$$g^{(n+1)}(0) \cdot x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g^{(n+1)}(t) e^{-xt} dt \geq 0 \tag{3.60}$$

for all real $x \neq 0$.

Proof. We have to consider the two cases $0 \in (\mathfrak{a}, \mathfrak{b})$ and $0 \in \{\mathfrak{a}, \mathfrak{b}\}$ separately.

1. Let $0 \in (\mathfrak{a}, \mathfrak{b})$. An application of Lemma 3.4.3 shows that equation (3.60) can be represented in $x \neq 0$ as

$$x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g^{(n+1)}(t) e^{-xt} dt = x^{n} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g_{(n)}(t) e^{-xt}, \tag{3.61}$$

   and the assumption $g^{(n)}(0) = 0$ shows that this identity can be written in the form

$$x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g^{(n+1)}(t) e^{-xt} dt = x^{n} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g_{(n-1)}(t) e^{-xt} dt,$$

   valid for $x \neq 0$. Lemma 3.5.4 proves the claim in this case.

2. Assume $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$. By assumption, g, g', and g'' are analytic on a set containing the real line and represent $F(z) \in \mathcal{E}$, $z^{-1}F(z) \in \mathcal{E}$, and $z^{-2}F(z) \in \mathcal{E}$, respectively. Thus, the condition $g^{(n)}(0) = 0$ implies $n \geq 3$ by Remark 3.3.13.

   Since $F(z)$ has a zero of order two at the origin, $z^{-2}F(z)$ has no zero at the origin, and the representation

$$z^{2}F^{-1}(z) = \mathcal{L}[g''](z)$$

   is valid in an open strip containing the origin. Using Lemma 3.4.3, the integral in (3.60) can be written as

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g^{(n+1)}(t) e^{-xt} dt &= \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) (g'')^{(n-1)}(t) e^{-xt} dt, \\ &= x^{n-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) e^{-xt} (g'')_{(n-2)}(t) dt.\end{aligned}$$

   The condition $g^{(n)}(0) = 0$ implies that $(g'')_{(n-2)}(t) = (g'')_{(n-3)}(t)$. We now apply (3.61) with $z^{-2}F(z) \in \mathcal{E}$, $n-2$, and $g''(t)$ instead of $F(z)$,

n, and $g(t)$ to finish the proof.

□

Lemma 3.5.6. *Let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1 with an $F \in \mathcal{E}$ which has a zero of multiplicity at least two at the origin. (In particular, $0 \in \{\mathfrak{a}, \mathfrak{b}\}$.) Then*

$$g'(0) \cdot x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g'(t) e^{-xt} dt > 0, \tag{3.62}$$

$$g''(0) \cdot x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) g''(t) e^{-xt} dt > 0 \tag{3.63}$$

for all real $x \neq 0$.

Proof. An integration by parts in $F^{-1}(z) = \mathcal{L}[g](z)$ yields

$$\frac{z}{F(z)} = \int_{-\infty}^{\infty} e^{-zt} g'(t) dt \text{ for } \mathfrak{a} < \Re z < \mathfrak{b}.$$

Since $z^{-1}F(z) \in \mathcal{E}$, Remark 3.3.13 is applicable, and thus the function g' is non-zero on $\mathbb{R}$ (hence does not change sign there). Since $x^{-1}\operatorname{sgn}(x,t) \geq 0$ for all real t and $x \neq 0$, inequality (3.62) follows.

Since $z^2 F^{-1}(z) \mathcal{L}[g''](z)$ and $z^{-2}F(z) \in \mathcal{E}$ by assumption, the same argument works for the second inequality. □

Lemma 3.5.7. *Let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1 with $0 \in [\mathfrak{a}, \mathfrak{b}]$. Let $c \neq 0$ be real, and assume that one of the conditions*

(i) $(D + c)g(0) = 0$,

(ii) $0 \in \{\mathfrak{a}, \mathfrak{b}\}$ *and* $e^{ct}g'(t)$ *is not integrable*

holds. Then $(D + c)g'(0) \neq 0$ and

$$(D + c)g'(0) \cdot x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) e^{-xt} (D + c) g'(t) dt > 0 \tag{3.64}$$

for $x \neq 0$.

Proof. First we assume (i), i.e. $(D + c)g(0) = 0$. Since $0 \in [\mathfrak{a}, \mathfrak{b}]$, every derivative of $f(t) := (D+c)g(t)$ is of polynomial growth by Lemma 3.2.9, and

Lemma 3.4.3 is applicable with $m = 0$. Since $(D+c)g(0) = 0$, $[(D+c)g]_(0) = (D+c)g$ and we see with equations (3.43) and (3.44) that

$$x^{-1}\int_{-\infty}^{\infty} \operatorname{sgn}(x,t)e^{-xt}D(D+c)g(t)dt = \int_{-\infty}^{\infty} \operatorname{sgn}(x,t)e^{-xt}(D+c)g(t)dt. \tag{3.65}$$

We can write the differential operator $D + c$ in the form $e^{-ct}De^{ct}$, i.e. $(D + c)g(t) = e^{-ct}De^{ct}g(t)$. The Laplace transform of $e^{ct}g(t)$ represents the reciprocal of $F(z+c) \in \mathcal{E}$ in some open vertical strip $S \subset \mathbb{C}$. We claim that the origin is in S. If this were not the case, then $e^{ct}g(t)$ would be monotone by Lemma 3.3.14. However, since

$$0 = (D + c)g(0) = e^{-ct}De^{ct}g(t)|_{t=0}, \tag{3.66}$$

we obtain $De^{ct}g(t)|_{t=0} = 0$. Lemma 3.3.14 implies that the origin is in S, and $e^{ct}g(t)$ is integrable by Remark 3.3.15.

Moreover, by Remark 3.2.5 the function $e^{ct}g(t)$ has no changes of sign, and by Corollary 3.3.12 the derivative $D[e^{ct}g(t)]$ has exactly one simple zero, which is at the origin by assumption.

This means that $(D + c)g(t) = e^{-ct}D[e^{ct}g(t)]$ changes sign only once, namely at the origin. Therefore by (3.58) for $m = 0$ the integrand on the right-hand side in (3.65) is of one sign for all real x and t. The sign of the integral in (3.64) equals the sign of $(D+c)g'(0)$, provided this value does not equal zero. It remains to show that strict inequality holds in (3.64).

By Corollary 3.3.12 every derivative of the function $e^{ct}g(t)$ has only simple zeros. Since $D[e^{ct}g(t)] = e^{ct}(D + c)g(t) = 0$ at the origin, we obtain $D^2e^{ct}g(t) \neq 0$ at $t = 0$. The identity

$$D(D + c)g(t) = De^{-ct}De^{ct}g(t) = e^{-ct}(D^2e^{ct}g(t) - cDe^{ct}g(t))$$

implies that $D(D + c)g(0) \neq 0$. Since $\operatorname{sgn}(x, t) > 0$ if $xt > 0$, the integral in (3.64) is non-zero on a set of positive measure, and inequality (3.64) follows with strict inequality.

Now assume that the assumptions (ii) hold. The condition $0 \in \{\mathfrak{a}, \mathfrak{b}\}$ means that $F(z)$ has a zero at the origin. We see that $z^{-1}F(z)$ is also an element of $\mathcal{E}$. From $zF^{-1}(z) = \mathcal{L}[g'](z)$ in $\mathfrak{a} < \Re z < \mathfrak{b}$ we obtain $(z +$

$c)F^{-1}(z+c) = \mathcal{L}[e^{ct}g'(t)](z)$. Therefore, Remark 3.3.15 is applicable to $e^{ct}g'(t)$. Since $e^{ct}g'(t)$ is not integrable by assumption, we see that $e^{ct}g'(t)$ must be monotone. We obtain that $De^{ct}g'(t)$ is of one sign and non-zero on $\mathbb{R}$. Since $x^{-1}\text{sgn}(x,t) \geq 0$ and $\text{sgn}(x,t) > 0$ for $xt > 0$a, the claim of (ii) follows. □

3.6 Majorants and Minorants

We are now in a position to construct entire majorants for certain classes of real polynomials.

Let $F \in \mathcal{E}$ be non-negative on the real line, and $F^{-1}(z) = \mathcal{L}[g](z)$ in a strip $\mathfrak{a} < \Re z < \mathfrak{b}$ with $0 \in [\mathfrak{a}, \mathfrak{b}]$. Recall the definition

$$G_P(z) = -\frac{F(z)}{z}\mathcal{L}[\text{sgn} \cdot P(D)g](z),$$

valid in the strip $\mathfrak{a} < \Re z < \mathfrak{b}$. By Theorem 3.5.2, the function $G_P(z)$ has an analytic continuation to the entire plane, if $F(0) = 0$ or $P(D)g(0) = 0$.

Let $n \in \mathbb{N}_0$. Lemmata 3.5.5 and 3.5.6 imply for the polynomial $P(z) = z^n$

Theorem 3.6.1. *Let $n \in \mathbb{N}$. Let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1 with $0 \in [\mathfrak{a}, \mathfrak{b}]$, and assume in addition that $F(z)$ satisfies $F \geq 0$ on the real line and either $0 \in (\mathfrak{a}, \mathfrak{b})$ or $F(z)$ has a zero of order two at the origin.*

If $g^{(n)}(0) = 0$, then the function $G_n(z)$ is entire, it satisfies

$$|G_n(x) - \text{sgn}(x)x^n| \ll x^{-2}|F(x)| \text{ as } x \to \pm\infty, \tag{3.67}$$

and

(i) if $g^{(n+1)}(0) < 0$, then $G_n(x) \geq \text{sgn}(x)x^n$ on $\mathbb{R}$,

(ii) if $g^{(n+1)}(0) > 0$, then $G_n(x) \leq \text{sgn}(x)x^n$ on $\mathbb{R}$.

Proof. Theorem 3.5.2 implies that G_P is entire and that (3.67) holds.

Since $F \geq 0$, the inequalities (i) and (ii) follow from Lemmata 3.5.3 and 3.5.5. □

Theorem 3.6.2. *Let $n \in \{0, 1\}$, and let F, g, G_P, $\mathfrak{a}$, and $\mathfrak{b}$ be as in Definition 3.5.1, and assume in addition that $F(z)$ has a zero of multiplicity two* *at the origin and satisfies $F \geq 0$. Then the function $G_n(z)$ is entire, satisfies*

$$|G_n(x) - \operatorname{sgn}(x)x^n| \ll |x|^{-2}|F(x)| \quad as \quad |x| \to \infty, \tag{3.68}$$

and

(i) the inequality $G_1(x) \leq |x|$ is valid on $\mathbb{R}$.

(ii) if $\mathfrak{a} = 0$, then $G_0(x) \leq \operatorname{sgn}(x)$ on $\mathbb{R}$,

(iii) if $\mathfrak{b} = 0$, then $G_0(x) \geq \operatorname{sgn}(x)$ on $\mathbb{R}$,

Proof. Theorem 3.5.2 implies that G_P is entire and that (3.68) holds.

Using integration by parts, the representatios $zF^{-1}(z) = \mathcal{L}[g'](z)$ and $z^2F^{-1}(z) = \mathcal{L}[g''](z)$ are shown to be valid in $\mathfrak{a} < \Re z < \mathfrak{b}$.

$z^{-1}F(z) \in \mathcal{E}$ and $z^{-2}F(z) \in \mathcal{E}$. Since $z^{-2}F(z) > 0$ on $(\mathfrak{a}, \mathfrak{b})$, $g'' \geq 0$, and (i) follows.

The condition $\mathfrak{a} = 0$ implies that $z^{-1}F(z) > 0$ on $(\mathfrak{a}, \mathfrak{b})$. Remark 3.2.6 implies that $g' \geq 0$, and (ii) follows.

Finally, we obtain that $\mathfrak{b} = 0$ implies $z^{-1}F(z) < 0$ on $(\mathfrak{a}, \mathfrak{b})$, which implies $g' \leq 0$. Property (iii) follows with another application of Lemma 3.5.6.

□

Let $\alpha \in [0, 1]$. In the following proposition we show that the choice

$$F_\alpha(z) := \pi^{-2} \sin^2 \pi(z - \alpha)$$

in Theorems 3.6.1 and 3.6.2 gives the extremal functions from Chapter 2. Recall the definition of the sequences α_n and β_n from Chapter 2:

$$\begin{array}{lllllll} \alpha_{4k} & := & 1 - z_{4k}, & \beta_{4k} & := & z_{4k}, \\ \alpha_{4k+1} & := & 0, & \beta_{4k+1} & := & \frac{1}{2}, \\ \alpha_{4k+2} & := & z_{4k+2}, & \beta_{4k+2} & := & 1 - z_{4k+2}, \\ \alpha_{4k+3} & := & \frac{1}{2}, & \beta_{4k+3} & := & 0. \end{array} \tag{3.69}$$

Define $g_\alpha(t) := te^{\alpha t}(e^t - 1)^{-1}$, and let

$$G_{\alpha,n}(z) := G_{F_\alpha, z^n}(z) = -\frac{F_\alpha(z)}{z}\mathcal{L}[\operatorname{sgn} \cdot g_\alpha^{(n+1)}](z).$$

Proposition 3.6.3. *With the above definitions we have*

$$G_{\alpha_n,n}(z) = H_n(z;\alpha_n),$$
$$G_{\beta_n,n}(z) = H_n(z;\beta_n),$$

where $H_n(z;\alpha_n)$ and $H_n(z;\beta_n)$ are the extremal minorants and majorants of $sgn(x)x^n$ from Theorem 2.5.3.

Proof. We start by noting the elementary identities

$$\sum_{k=0}^{N-1} \frac{1}{(z-k)^2} = -\sum_{k=0}^{N-1} \int_{-\infty}^{0} te^{-(z-k)t} dt = -\int_{-\infty}^{0} te^{-zt} \frac{e^{Nt}-1}{e^t-1} dt \text{ for } \Re z < 0,$$
$$\sum_{k=1}^{N} \frac{1}{(z+k)^2} = \sum_{k=1}^{N} \int_{0}^{\infty} te^{-(z+k)t} dt = \int_{0}^{\infty} te^{-zt} e^{-t} \frac{e^{-Nt}-1}{e^{-t}-1} dt \text{ for } \Re z > -1.$$

Letting $N \to \infty$, we obtain with Lebesgue's theorem of dominated convergence

$$\sum_{k=0}^{\infty} \frac{1}{(z-k)^2} = \int_{-\infty}^{0} e^{-zt} \frac{t}{e^t-1} dt \text{ for } \Re z < 0,$$
$$-\sum_{k=-\infty}^{-1} \frac{1}{(z-k)^2} = -\int_{0}^{\infty} e^{-zt} \frac{t}{e^t-1} dt \text{ for } \Re z > -1.$$

Replacing z by $z-\alpha$ we obtain with $\mathrm{sgn}_+(x) := \mathrm{sgn}(x+)$

$$\sum_{k=-\infty}^{\infty} \frac{\mathrm{sgn}_+(k)}{(z-k-\alpha)^2} = -\int_{-\infty}^{\infty} \mathrm{sgn}(t) \frac{te^{\alpha t}}{e^t-1} e^{-zt} dt \text{ for } \alpha - 1 < \Re z < \alpha. \quad (3.70)$$

Using (3.70) in Definition 2.5.1 we obtain in the region $\alpha - 1 < \Re z < \alpha$

$$\begin{aligned}
H_n(z;\alpha) &= F_\alpha(z)\Big(z^n \sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}_+(k)}{(z-k-\alpha)^2} + 2\sum_{k=1}^{n} B_{k-1}(\alpha) z^{n-k} + 2\frac{B_n(\alpha)}{z-\{\alpha\}}\Big)\\
&= F_\alpha(z)\Big(-z^n \int_{-\infty}^{\infty} \operatorname{sgn}(t)\frac{te^{\alpha t}}{e^t-1} e^{-zt}dt + 2\sum_{\mu=0}^{n-1} B_\mu(\alpha) z^{n-\mu-1}\\
&\qquad + 2\frac{B_n(\alpha)}{z-\{\alpha\}}\Big)\\
&= -\frac{F_\alpha(z)}{z}\mathcal{L}[\operatorname{sgn}\cdot g^{(n+1)}](z) + 2B_n(\alpha)F_\alpha(z)((z-\{\alpha\})^{-1} - z^{-1}).
\end{aligned} \tag{3.71}$$

Here we used Lemma 3.4.1 applied to $f(t) = te^{\alpha t}(e^t-1)^{-1}$ which has the power series expansion

$$\frac{te^{\alpha t}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k(\alpha)}{k!} t^k. \tag{3.72}$$

Recall equation (3.19):

$$\Big(\frac{\pi}{\sin \pi(z-\alpha)}\Big)^2 = \int_{-\infty}^{\infty} e^{-zt}\frac{te^{\alpha t}}{e^t-1}dt \text{ for } \alpha - 1 < \Re z < \alpha. \tag{3.73}$$

Thus, (3.71) becomes for $\alpha - 1 < \Re z < \alpha$

$$H_n(z;\alpha) = G_{\alpha,n}(z) + 2B_n(\alpha)F_\alpha(z)((z-\{\alpha\})^{-1} - z^{-1}). \tag{3.74}$$

The second term on the right-hand side of (3.74) equals zero for the choices $\alpha = \alpha_n$ and $\alpha = \beta_n$ defined in (3.69). Since Proposition 3.6.3 holds in $\alpha - 1 < \Re z < \alpha$ and since $H_n(z;\alpha)$ is an entire function, the claim follows by analytic continuation. □

Let $P(x) := ax + b$ where $a \neq 0$ and b are real numbers. From Theorem 2.5.3 we obtain majorants for $a|x|$ and $b\operatorname{sgn}(x)$, and the sumof these majorants will be a majorant of $\operatorname{sgn}(x)P(x)$. However, for most linear real polynomials a majorant obtained in this way is not extremal. In the following corollary we give the extremal entire majorants of type $\delta = 2\pi$ for the function $\operatorname{sgn}(x)P(x)$.

Recall

$$F_\alpha(z) := \pi^{-2} \sin^2 \pi(z-\alpha).$$

We use the notation

$$G_{\alpha,P}(z) := -\frac{F_\alpha(z)}{z}\mathcal{L}[\mathrm{sgn}\cdot(aD+b)g'_\alpha](z), \tag{3.75}$$

where $g_\alpha(t) = (e^t-1)^{-1}te^{\alpha t}$. Furthermore, let $c := a/b$ and define

$$\alpha_P := \begin{cases} 1 & \text{if } c \geq 0 \\ 0 & \text{if } c < 0 \end{cases} \tag{3.76}$$

$$\tag{3.77}$$

(as we shall see below, the choice $\alpha_P = 1$ for $c = 0$ is not essential; $\alpha_P = 0$ would lead to the same results), and

$$\beta_P := \begin{cases} 0 & \text{if } c > 1/2 \\ 1/2 - c & \text{if } -1/2 < c \leq 1/2 \\ 1 & \text{if } c \leq -1/2. \end{cases} \tag{3.78}$$

Corollary 3.6.4. *Set $P(x) := ax + b$ where $a \neq 0$ and b are real numbers. Define $c := a/b$. If $a > 0$, then*

$$G_{\alpha_P,P}(x) \leq \mathrm{sgn}(x)P(x) \leq G_{\beta_P,P}(x)$$

If $a < 0$, the inequalities in this corollary are reversed. Moreover, we have

$$\int_{-\infty}^{\infty}(G_{0,P}(x) - \mathrm{sgn}(x)P(x))dx = -\frac{a}{6} + b$$
$$\int_{-\infty}^{\infty}(G_{1,P}(x) - \mathrm{sgn}(x)P(x))dx = -\frac{a}{6} - b$$
$$\int_{-\infty}^{\infty}(G_{1/2-c,P}(x) - \mathrm{sgn}(x)P(x))dx = a/12 + b^2/a.$$

The functions $G_{\alpha_P,P}(x)$ and $G_{\beta_P,P}(x)$ are the extremal minorant and majorant, respectively, of $\mathrm{sgn}(x)P(x)$.

Here we give proofs for all cases except majorants with $1/2 < |c| < 1$; the proofs in the remaining ranges are analogous, but as the arguments now are structured, require changes of details and notation throughout Chapter 3.

Proof. Assume first that $a = 1$, i.e. $b = c$. Recall the definition $F_\alpha(z) := \pi^{-2}\sin^2\pi(z-\alpha)$. We shall apply Lemmata 3.5.3 and 3.5.7 to $G_{\alpha,P}(z)$. By (3.73) we have for any $\alpha \in \mathbb{R}$

$$F_\alpha^{-1}(z) = \mathcal{L}[g_\alpha](z) \text{ for } \alpha - 1 < \Re z < \alpha, \tag{3.79}$$

where $g_\alpha(t) := te^{\alpha t}(e^t-1)^{-1}$. In order to apply Lemma 3.5.7 we have to investigate

$$\begin{aligned}(D+c)g_\alpha(0) &= g_\alpha'(0) + cg_\alpha(0),\\ (D+c)g_\alpha'(0) &= g_\alpha''(0) + cg_\alpha'(0).\end{aligned}$$

where $\alpha = \alpha_P$ or $\alpha = \beta_P$. By (3.72) we have

$$\begin{aligned}(D+c)g_\alpha(0) &= B_1(\alpha) + cB_0(\alpha) = \alpha - \frac{1}{2} + c\\ (D+c)g_\alpha'(0) &= B_2(\alpha) + cB_1(\alpha) = \alpha^2 - \alpha + \frac{1}{6} + c\alpha - \frac{c}{2}.\end{aligned}$$

First we prove the majorants for $-1/2 \le c \le 1/2$. Recall $\beta_P = 1/2 - c$ in this case. We shall show that Lemma 3.5.7 (i) applies. From (3.79) we see that $\mathfrak{a} = 1/2 - c - 1 \le 0$ and $\mathfrak{b} = 1/2 - c \ge 0$, so the condition $0 \in [\mathfrak{a}, \mathfrak{b}]$ holds. Moreover, since

$$(D+c)g_{1/2-c}(0) = \frac{1}{2} - c - \frac{1}{2} + c = 0,$$

Lemma 3.5.7 (i) applies. Since

$$\begin{aligned}(D+c)g_{1/2-c}'(0) &= \Big(\frac{1}{2}-c\Big)^2 - \Big(\frac{1}{2}-c\Big) + \frac{1}{6} + c\Big(\frac{1}{2}-c\Big) - \frac{c}{2}\\ &= \frac{1}{4} - c + c^2 - \frac{1}{2} + c + \frac{1}{6} + \frac{c}{2} - c^2 - \frac{c}{2} = -\frac{1}{12} < 0,\end{aligned} \tag{3.80}$$

we obtain from Lemma 3.5.3, (3.80), and Lemma 3.5.7

$$\begin{aligned}&G_{1/2-c,P}(x) - \operatorname{sgn}(x)P(x)\\ &\qquad = -F_{1/2-c}(x)x^{-1}\int_{-\infty}^{\infty}\operatorname{sgn}(x,t)e^{-xt}(D+c)g_{1/2-c}'(t)dt \ge 0.\end{aligned}$$

This shows the majorants for $-1/2 \le c \le 1/2$.

Now we consider the minorants for $c \ge 0$. Recall $\alpha_P = 1$ in this case. For this value of α, (3.79) holds in the strip $0 < \Re z < 1$, i.e. $\mathfrak{a} = 0$ in Lemma 3.5.7. We have

$$e^{ct} g_1'(t) = e^{ct} \frac{d}{dt} t e^t (e^t - 1)^{-1} = e^{ct}\Big((1+t)\frac{e^t}{e^t - 1} - t\frac{e^{2t}}{(e^t-1)^2}\Big) \sim e^{ct}$$

as $t \to \infty$. Therefore $e^{ct} g_1'(t)$ is not integrable for $c \ge 0$, and Lemma 3.5.7 (ii) applies. Since

$$(D+c) g_1'(0) = 1 - 1 + \frac{1}{6} + c - \frac{c}{2} = \frac{c}{2} + \frac{1}{6} > 0,$$

we obtain for $x \neq 0$ from Lemma 3.5.3 and Lemma 3.5.7

$$G_{1,P}(x) - \operatorname{sgn}(x) P(x) = -\Big(\frac{\sin \pi x}{\pi}\Big)^2 x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) e^{-xt} (D+c) g_1'(t) dt \le 0.$$

Now we consider the minorants for $c \le 0$. Recall $\alpha_P = 0$ for polynomials P with $c \le 0$. For this value of α, (3.79) holds in the strip $-1 < \Re z < 0$, i.e. $\mathfrak{b} = 0$ in Lemma 3.5.7. We have

$$e^{ct} g_0'(t) = e^{ct}\Big(\frac{1}{e^t - 1} - \frac{t e^t}{(e^t - 1)^2}\Big) \sim -e^{ct}$$

as $t \to -\infty$. Therefore $e^{ct} g_0'(t)$ is not integrable for $c \le 0$, and Lemma 3.5.7 (ii) applies. Since

$$(D+c) g_0'(0) = \frac{1}{6} - \frac{c}{2} > 0$$

we obtain

$$G_{0,P}(x) - \operatorname{sgn}(x) P(x) = -\Big(\frac{\sin \pi x}{\pi}\Big)^2 x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t) e^{-xt} (D+c) g_0'(t) dt \le 0,$$

which proves the minorants in the third and fourth case.

Next we consider the majorants for $c \ge 1$. Recall $\beta_P = 0$ for $c \ge 1$. We have

$$e^{ct} g_0'(t) = e^{ct}\Big(\frac{1}{e^t - 1} - \frac{t e^t}{(e^t - 1)^2}\Big) \sim \frac{t e^{(c+1)t}}{(e^t - 1)^2}$$

as $t \to \infty$. Therefore $e^{ct} g_0'(t)$ is not integrable, and Lemma 3.5.7 (ii) applies.

Since

$$(D+c)g_0'(0) = \frac{1}{6} - \frac{c}{2} \le \frac{1}{6} - \frac{1}{2} < 0,$$

we obtain

$$G_{0,P}(x) - \operatorname{sgn}(x)P(x) = -\Big(\frac{\sin \pi x}{\pi}\Big)^2 x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t)e^{-xt}(D+c)g_0'(t)dt \ge 0,$$

which proves the majorants in the case $c \ge 1$.

Finally, let $c \le -1$ and recall $\beta_P = 1$ for $c \le -1$. In this case

$$e^{ct}g_1'(t) = e^{ct}(1+t)\Big(\frac{e^t}{e^t-1} - t\frac{e^{2t}}{(e^t-1)^2}\Big) \sim -te^{(c+1)t}$$

as $t \to -\infty$, and thus $e^{ct}g_1'(t)$ is not integrable. Since

$$(D+c)g_1'(0) = 1 - 1 + \frac{1}{6} + c - \frac{c}{2} \le -\frac{1}{2} + \frac{1}{6} < 0,$$

we obtain

$$G_{1,P}(x) - \operatorname{sgn}(x)P(x) = -\Big(\frac{\sin \pi x}{\pi}\Big)^2 x^{-1} \int_{-\infty}^{\infty} \operatorname{sgn}(x,t)e^{-xt}(D+c)g_1'(t)dt \ge 0,$$

which proves the majorants in the case $c \le -1$.

If $a \ne 1$, we consider the polynomial $x + b/a$, and multiply the inequalities that we get by a. Since $G_{\alpha,P}(z)$ contains the differential operator $aD + b$ in its definition, we obtain

$$aG_{\alpha,z+b/a}(x) = G_{\alpha,az+b}(x),$$

and in particular $-G_{\alpha,-P}(x) = G_{\alpha,P}(x)$. Since $c = b/a = (-b)/(-a)$, it follows that

$$\alpha_{-P} = \alpha_P \text{ and } \beta_{-P} = \beta_P. \tag{3.81}$$

Thus, the majorant of P for $a < 0$ can be obtained in the following way: we construct the minorant $G_{\alpha_{-P},-P}(x) \le \operatorname{sgn}(x)(-P(x))$, which is equal to $G_{\alpha_P,-P}(x)$ by (3.81). It follows that $G_{\alpha_P,P}(x) = -G_{\alpha_P,-P}(x) \ge \operatorname{sgn}(x)P(x)$.

Next we show the extremality of the majorants and minorants in Corollary 3.6.4. By Theorem 2.5.3 there exist entire majorants of type 2π for $a|x|$ and

$b\,\mathrm{sgn}(x)$. Their sum is an entire majorant of type 2π for $\mathrm{sgn}(x)P(x)$. (In general this majorant is not extremal, though.)

Assume now that $F(x)$ is entire of type 2π, that it satisfies the inequality $F(x) \geq \mathrm{sgn}(x)P(x)$ for real x, and that $\int_{\mathbb{R}}(F(x)-\mathrm{sgn}(x)P(x))dx < \infty$ holds. Set

$$\psi(x) := F(x) - \mathrm{sgn}(x)P(x).$$

The difference $\psi - a\psi_{1,0} - b\psi_{0,0}$ is an integrable entire function of type 2π. It follows from Theorem 2.2.1 and Lemma 2.5.5 that

$$\widehat{\psi}(t) = -a\frac{2\cdot 1!}{(2\pi it)^2} - b\frac{2\cdot 0!}{(2\pi it)^1} = -\frac{2a}{(2\pi it)^2} - \frac{2b}{2\pi it} \text{ for } |t| \geq 1$$

holds. An argument similar to the one given in the proof of Lemma 2.5.6 shows for $x \in (0,1)$ with Lemma 2.3.1

$$\begin{aligned} 0 \leq \sum_{m\in\mathbb{Z}} \psi(m+x) &= \widehat{\psi}(0) - 2a\sum_{k\neq 0}\frac{e(kx)}{(2\pi ik)^2} - 2b\sum_{k\neq 0}\frac{e(kx)}{2\pi ik} \\ &= \widehat{\psi}(0) + a\mathcal{B}_2(x) + 2b\mathcal{B}_1(x). \end{aligned}$$

It follows that

$$\widehat{\psi}(0) \geq -\min_{0\leq x\leq 1}(aB_2(x) + 2bB_1(x)). \tag{3.82}$$

By equation (3.74) and (3.75) we have for $\alpha \in [0,1]$

$$\begin{aligned} &G_{\alpha,P}(x) - aH_1(x;\alpha) + bH_0(x;\alpha) = \\ &\qquad - 2F_\alpha(x)(aB_1(\alpha) + bB_0(\alpha)((z-\{\alpha\})^{-1} - z^{-1}), \end{aligned} \tag{3.83}$$

and we claim that the right-hand side of (3.83) equals zero for the values $\alpha = \alpha_P$ and $\alpha = \beta_P$. For $\alpha_P, \beta_P \in \{0,1\}$ this is evident. If $c = b/a \in [-1/2, 1/2]$, then $\beta_P = 1/2 - c$, and

$$aB_1(\beta_P) + bB_0(\beta_P) = a\Big(\frac{1}{2} - c - \frac{1}{2}\Big) + b = -b + b = 0,$$

and the right-hand side of (3.83) equals zero in this case as well. We obtain for $\alpha \in \{\alpha_P, \beta_P\}$

$$\begin{aligned}
\int_{-\infty}^{\infty} &(G_{\alpha,P}(x) - \operatorname{sgn}(x)P(x))dx \\
&= \int_{-\infty}^{\infty} (aH_1(x;\alpha) + bH_0(x;\alpha) - \operatorname{sgn}(x)(ax+b))dx \\
&= -aB_2(\alpha) - 2bB_1(\alpha) \qquad (3.84)
\end{aligned}$$

by Theorem 2.5.3. Equations (3.82) and (3.84) show the extremality of the majorants in Corollary 3.6.4. The extremality of the minorants follows similarly.

It remains to verify the claimed values in Corollary 3.6.4. In (3.84) we have shown that

$$\int_{-\infty}^{\infty} (G_{\alpha,P}(x) - \operatorname{sgn}(x)P(x))dx = -aB_2(\alpha) - 2bB_1(\alpha)$$

for $\alpha \in \{\alpha_P, \beta_P\}$ and $\alpha \in \{0, 1\}$. We have

$$aB_2(0) + 2bB_1(0) = a\Big(\alpha^2 - \alpha + \frac{1}{6}\Big) + b\Big(\alpha - \frac{1}{2}\Big).$$

Therefore

$$\begin{aligned}
aB_2(0) + 2bB_1(0) &= \frac{a}{6} - b \\
aB_2(1) + 2bB_1(1) &= \frac{a}{6} + b.
\end{aligned}$$

We obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} (G_{0,P}(x) - \operatorname{sgn}(x)P(x))dx &= -\frac{a}{6} + b, \\
\int_{-\infty}^{\infty} (G_{1,P}(x) - \operatorname{sgn}(x)P(x))dx &= -\frac{a}{6} - b,
\end{aligned}$$

which proves the first and second equalities in Corollary 3.6.4. If $c = b/a \in$

$(-1/2, 1/2)$ then $\beta_P = 1/2 - c$. Since

$$\begin{aligned} &aB_2(1/2-c) + 2bB_1(1/2-c) \\ &\qquad\qquad = a\Big(\frac{1}{4} - c + c^2\Big) - a\Big(\frac{1}{2} - c\Big) + a\frac{1}{6} + 2b\Big(\frac{1}{2} - c - \frac{1}{2}\Big) \\ &\qquad\qquad = -\frac{a}{12} - \frac{b^2}{a}, \end{aligned}$$

the third equality in Corollary 3.6.4 follows. □

The next page contains plots of $\operatorname{sgn}(x)P(x)$ and $G_{\alpha,P}(x)$. We define

$$P_c(x) := x + c.$$

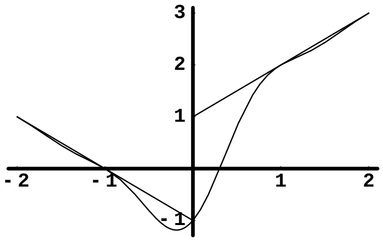

Figure 3.1: P_1 and $G_{1,P}$

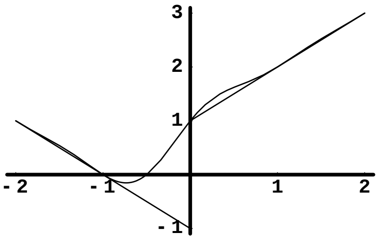

Figure 3.2: P_1 and $G_{0,P}$

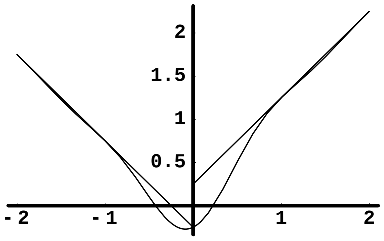

Figure 3.3: $P_{1/4}$ and $G_{1,P}$

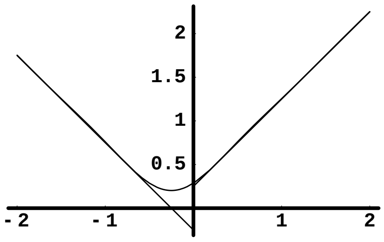

Figure 3.4: $P_{1/4}$ and $G_{1/4,P}$

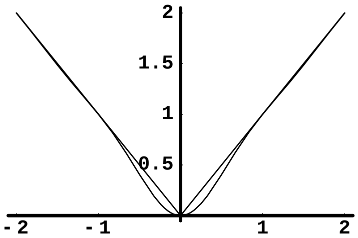

Figure 3.5: P_0 and $G_{0,P}$

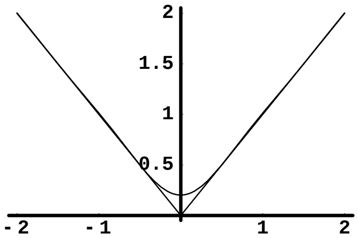

Figure 3.6: P_0 and $G_{1/2,P}$

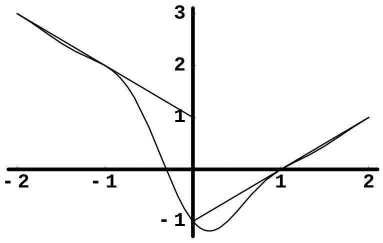

Figure 3.7: P_{-1} and $G_{0,P}$

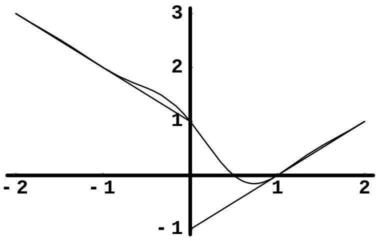

Figure 3.8: P_{-1} and $G_{1,P}$

Chapter 4

Applications

4.1 Bounded Functions with a Spectral Gap

Let $m \in \mathbb{N}$, $\delta > 0$, and $\alpha \in [0,1]$. By Lemma 2.5.5 the functions

$$\varphi_{\delta,m,\alpha}(x) := \frac{1}{2(m-1)!}(\operatorname{sgn}(x)x^{m-1} - \delta^{-m+1}H_{m-1}(\delta x;\alpha)) \tag{4.1}$$

are integrable and satisfy

$$\widehat{\varphi}_{\delta,m,\alpha}(t) = (2\pi i t)^{-m} \text{ for } |t| \geq \delta. \tag{4.2}$$

For $f \in L^2(\mathbb{R})$ with a square integrable mth derivative the relation

$$\widehat{f}(t) = (2\pi i t)^{-m}\widehat{f^{(m)}}(t) \tag{4.3}$$

is valid. Consider functions f with the additional condition

$$\operatorname{supp}\widehat{f} \cap [-\delta,\delta] = \emptyset. \tag{4.4}$$

Equations (4.2) and (4.3) suggest that the functions $\varphi_{\delta,m,\alpha}$ are mth integrating kernels for functions f satisfying (4.4), i.e.

$$f = \varphi_{\delta,m,\alpha} * f^{(m)}. \tag{4.5}$$

We show this result for functions which have a bounded, locally integrable mth derivative. In order to use (4.2) for non-integrable functions, we introduce the notion of distributions of slow growth and their Fourier transforms.

The notation and results are taken from [26].

We consider the testing space $\mathcal{S}$ of all functions $\theta : \mathbb{R} \to \mathbb{C}$ that are infinitely smooth and are such that, as $|t| \to \infty$, they and all their partial derivatives decrease to zero faster than any power of $1/|t|$. $\mathcal{S}$ is a linear space, and any derivative of a function in $\mathcal{S}$ is again an element of $\mathcal{S}$.

A continuous linear functional on $\mathcal{S}$ is called a tempered distribution. We denote by $\mathcal{S}'$ the space of all tempered distributions. If a functional F in $\mathcal{S}'$ is given by $\theta \mapsto \langle F, \theta \rangle = \int_{\mathbb{R}} f\theta$ with a function $f : \mathbb{R} \to \mathbb{C}$, we abuse notation by saying that $f \in \mathcal{S}'$.

From [26] Section 4.3 we have the following: in order for a locally integrable function $f(t)$ to assign a finite number $\langle f, \theta \rangle$ to every testing function $\theta \in \mathcal{S}$ the behavior of f at $\pm\infty$ must be so restricted that the integral $\int_{\mathbb{R}} f(t)\theta(t)dt$ converges for all $\theta \in \mathcal{S}$. This is certainly justified if the function f satisfies

$$f(t) \ll (1 + |t|)^N \tag{4.6}$$

on the real line for some positive integer N.

Let $F \in \mathcal{S}'$. We define the distributional Fourier transform $\widehat{F}$ of the functional F by means of a Parseval-type relation, namely $\widehat{F}$ is defined as the unique functional which satisfies

$$\langle \widehat{F}, \theta \rangle = \langle F, \widehat{\theta} \rangle \text{ for all } \theta \in \mathcal{S}, \tag{4.7}$$

where the Fourier transform $\widehat{\theta}$ is defined in the usual sense, cf. (1.18). (Note that the Fourier transform $\widehat{\theta}$ of $\theta \in \mathcal{S}$ satisfies $\widehat{\theta} \in \mathcal{S}$.)

Let α_n, β_n be given by (2.33). Using (4.1), define

$$\varphi^{+}_{\delta,m} := \varphi_{\delta,m,\alpha_{m-1}}, \tag{4.8}$$

$$\varphi^{-}_{\delta,m} := \varphi_{\delta,m,\beta_{m-1}}, \tag{4.9}$$

and define for $0 \leq \alpha \leq 1$ the δ-normalized nth Bernoulli polynomial by

$$B_{n,\delta}(\alpha) := \frac{B_n(\alpha)}{\delta^n n!}. \tag{4.10}$$

where $B_n(\alpha)$ is the nth Bernoulli polynomial introduced in Section 2.3.

The functions $\varphi_{\delta,m,\alpha}$ are integrating kernels for functions with a spectral gap containing the origin in the following sense:

Proposition 4.1.1. *Let $m \in \mathbb{N}$, let $\delta > 0$, and let $f \in \mathcal{S}'$ be a real-valued function with a bounded* m*th derivative and whose distributional Fourier transform satisfies* supp $\widehat{f} \cap [-\delta, \delta] = \emptyset$. *Then for any $\alpha \in [0, 1]$*

$$f = f^{(m)} * \varphi_{\delta,m,\alpha}. \tag{4.11}$$

Moreover, $\int_{\mathbb{R}} \varphi_{\delta,m,\alpha}(x)dx = B_{m,\delta}(\alpha)$, $\varphi^{+}_{\delta,m} \geq 0$, and $\varphi^{-}_{\delta,m} \leq 0$.

Proof. Let $\theta \in \mathcal{S}$. Since $f^{(m)}$ is bounded and locally integrable, the convolution $f^{(m)} * \varphi_{m,\delta,\alpha}$ is bounded and locally integrable as well. In particular, $f^{(m)} * \varphi_{m,\delta,\alpha}$ satisfies (4.6), i.e. $f^{(m)} * \varphi_{m,\delta,\alpha} \in \mathcal{S}'$. Since for any $F \in \mathcal{S}'$ the Fourier transform satisfies $\widehat{F} \in \mathcal{S}'$, we have that

$$[f^{(m)} * \varphi_{m,\delta,\alpha}]^{\wedge} \in \mathcal{S}'. \tag{4.12}$$

By Lemma 2.5.5, equation (2.49), we have $\widehat{\varphi}_{\delta,m,\alpha}(t) = (2\pi i t)^{-m}$ for $|t| \geq \delta$. We obtain for any $f \in \mathcal{S}'$ with supp $\widehat{f} \cap [-\delta, \delta] = \emptyset$

$$\langle \widehat{f}, \theta \rangle = \left\langle \frac{\widehat{f^{(m)}}}{(2\pi i t)^m}, \theta \right\rangle = \langle \widehat{f^{(m)}} \widehat{\varphi}_{\delta,m,\alpha}, \theta \rangle,$$

and (4.12) implies

$$f = f^{(m)} * \varphi_{\delta,m,\alpha}.$$

The last three statements of the proposition are direct consequences of Theorem 2.5.3 applied to $m = n + 1$. □

Next we prove a generalization of a result of Beurling found in [25], Theorem 15.

Theorem 4.1.2. *Let $m \in \mathbb{N}$, and let $f \in \mathcal{S}'$ be real-valued with a bounded* m*th derivative such that* supp $\widehat{f} \cap [-\delta, \delta] = \emptyset$ *for some $\delta > 0$. Then*

$$|f(x)| \leq \max\{B_{m,\delta}(\alpha_{m-1}), -B_{m,\delta}(\beta_{m-1})\} \sup_{x \in \mathbb{R}} f^{(m)}(x). \tag{4.13}$$

Remark. By replacing f by $-f$ we see that the supremum in Theorem 4.1.2 can be replaced by the negative of the infimum.

Proof of Theorem 4.1.2. Let $f \in \mathcal{S}'$ be real-valued, have a bounded mth derivative, and satisfy supp $\widehat{f} \cap [-\delta, \delta] = \emptyset$. Since $\varphi^{+}_{\delta,m-1} \geq 0$ we have

$$\begin{aligned} f(x) = f^{(m)} * \varphi^{+}_{\delta,m}(x) &\leq \sup_{x\in\mathbb{R}} f^{(m)}(x) \int_{\mathbb{R}} \varphi^{+}_{\delta,m}(u)dt \\ &= B_{m,\delta}(\alpha_{m-1}) \sup_{x\in\mathbb{R}} f^{(m)}(x); \end{aligned}$$

and since $\varphi^{-}_{\delta,m} \leq 0$ we have

$$\begin{aligned} -f(x) = f^{(m)} * (-\varphi^{-}_{\delta,m})(x) &\leq \sup_{x\in\mathbb{R}} f^{(m)}(x) \int_{\mathbb{R}} (-\varphi^{-}_{\delta,m}(u))dt \\ &= -B_{m,\delta}(\beta_{m-1}) \sup_{x\in\mathbb{R}} f^{(m)}(x), \end{aligned}$$

which proves the theorem. □

4.2 Positive Definite Extensions into the Unit Interval

Let S be $\mathbb{R}$ or $\mathbb{Z}$. We say that a function $f : S \to \mathbb{C}$ is *positive definite* if for every $N \in \mathbb{N}$, any $a_1, ..., a_N \in \mathbb{C}$, and any $x_1, ..., x_N \in S$ the inequality

$$\sum_{\nu,\mu=1}^{N} a_\nu \overline{a}_\mu f(x_\nu - x_\mu) \geq 0 \tag{4.14}$$

holds.

Let $m \in \mathbb{N}$. As a corollary of Theorem 2.5.3 we obtain positive definite extensions to $\mathbb{R}$ of the functions $\pm m!(2\pi i t)^{-m}$ restricted to $\mathbb{R}\backslash[-1, 1]$. Define

$$s_{m,\alpha}(t) := -2 \sum_{k=0}^{\infty} \frac{B_{k+m}(\alpha)}{(k+1)!} \Big(\frac{k+1}{k+m} - |t|\Big)(-2\pi i t)^k \tag{4.15}$$

where $0 \leq \alpha \leq 1$, $m \in \mathbb{N}$, and $|t| < 1$.

Corollary 4.2.1. *Let $m \in \mathbb{N}$. The following functions are positive definite* *on $\mathbb{R}$:*

$$f_m(t) = \begin{cases} m!(2\pi it)^{-m} & \textit{if } |t| \geq 1, \\ s_{m,\alpha_{m-1}}(t) & \textit{if } |t| < 1, \end{cases} \tag{4.16}$$

$$g_m(t) = \begin{cases} -m!(2\pi it)^{-m} & \textit{if } |t| \geq 1, \\ -s_{m,\beta_{m-1}}(t) & \textit{if } |t| < 1. \end{cases} \tag{4.17}$$

The following functions are positive definite on $\mathbb{Z}$:

$$p_m(k) = \begin{cases} m!(2\pi ik)^{-m} & \textit{if } k \neq 0, \\ B_m(\alpha_{m-1}) & \textit{if } k = 0. \end{cases} \tag{4.18}$$

$$q_m(k) = \begin{cases} -m!(2\pi ik)^{-m} & \textit{if } k \neq 0, \\ -B_m(\beta_{m-1}) & \textit{if } k = 0. \end{cases} \tag{4.19}$$

Moreover, $f_m(0) = B_m(\alpha_{m-1})$, $g_m(0) = -B_m(\beta_{m-1})$, and the values $f_m(0)$, $g_m(0)$, $p_m(0)$, $q_m(0)$ are all minimal in the sense that none of the functions $\pm m!(2\pi it)^{-m}$ (resp. $\pm(2\pi ik)^{-m}$) restricted to $\mathbb{R}\backslash[-1,1]$ (resp. $\mathbb{Z}\backslash\{0\}$) can have a positive extension to $\mathbb{R}$ (resp. $\mathbb{Z}$) having a smaller value at the origin.

Proof. We prove statements (4.16) and (4.18). Let $n \in \mathbb{N}_0$. By Theorem 2.5.3

$$\phi_{n,\alpha_n}(x) = \operatorname{sgn}(x)x^n - H_n(x;\alpha_n) \geq 0,$$

by Lemma 2.5.4 the function is integrable on $\mathbb{R}$, and by Lemma 2.5.5

$$\widehat{\phi}_{n,\alpha_n}(t) = \frac{2 \cdot n!}{(2\pi it)^{n+1}}$$

for $|t| \geq 1$. By the easy implication of Bochner's theorem, $\widehat{\phi}_{n,\alpha_n}$ is positive definite. Lemma 2.5.5 (2.48) yields the explicit representation of $\widehat{\phi}_{n,\alpha_n}(t)$ for $|t| < 1$. Note that the last term in (2.48) is equal to zero, since by definition one of the equations $B_n(\alpha_n) = 0$, $\alpha_n = 0$, or $\alpha_n = 1$ holds. Performing the substitution $m = n+1$ yields (4.16).

For the proof of (4.18) consider the function $p_{m,c} : \mathbb{Z} \to \mathbb{C}$ $(c \in \mathbb{R})$

defined by

$$p_{m,c}(k) = \begin{cases} m!\,(2\pi i k)^{-m} & \text{if } k \neq 0 \\ c & \text{if } k = 0 \end{cases} \qquad (k \in \mathbb{Z}).$$

By (2.3.1)

$$\sum_{k\in\mathbb{Z}} p_{m,c}(k)e(kt) = c - \mathcal{B}_m(t),$$

and this is non-negative if, and only if,

$$c \geq \mathcal{B}_m(t) \text{ for all } t \in [0,1].$$

We obtain using Bochner's theorem that $p_{m,c}$ is a positive definite function on $\mathbb{Z}$ if, and only if, $c \geq \max \mathcal{B}_m(t) = B_m(\alpha_{m-1})$, which shows that $B_m(\alpha_{m-1})$ is the minimal value at $k = 0$ which extends $(2\pi i k)^{-m}$ $(k \neq 0)$ to a positive definite function on $\mathbb{Z}$. This shows (4.18).

Moreover, if $c < B_m(\alpha_{m-1})$, then there exist $N \in \mathbb{N}$, numbers $a_\nu \in \mathbb{C}$, and distinct numbers $\lambda_\nu \in \mathbb{Z}$ such that

$$\sum_{\nu,\mu=1}^{N} a_\nu \overline{a}_\mu f(\lambda_\nu - \lambda_\mu) = \sum_{\nu,\mu=1}^{N} a_\nu \overline{a}_\mu p_{m,c}(\lambda_\nu - \lambda_\mu) < 0, \tag{4.20}$$

which shows that the value $f_m(0) = B_m(\alpha_{m-1})$ in (4.16) is optimal as well.

Statements (4.17) and (4.19) follow similarly by considering

$$\psi_{n,\beta_n}(x) = H_n(x;\beta_n) - \operatorname{sgn}(x)x^n.$$

instead of ϕ_{n,α_n}. $\square$

4.3 Hilbert-type Inequalities

As a consequence of Corollary 4.2.1 we obtain sharp bounds in certain Hilbert type inequalities. Let $(a_\nu)_{\nu=1}^N$ be a finite sequence of complex numbers, and let $\{\lambda_\nu\}_{\nu=1}^N$ be a set of real numbers which are well-spaced in the sense that $|\lambda_\nu - \lambda_\mu| \geq \delta > 0$ for all $\nu \neq \mu$, and let $h(t)$ $(t \in \mathbb{R})$ be a hermitian function, i.e. $h(-t) = \overline{h(t)}$. We are interested in optimal bounds $L_\delta(h)$ and $U_\delta(h)$ such that

$$-L_\delta(h)\sum_{\nu=1}^N |a_\nu|^2 \le \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^N a_\nu\overline{a}_\mu h(\lambda_\nu-\lambda_\mu) \le U_\delta(h)\sum_{\nu=1}^N |a_\nu|^2 \tag{4.21}$$

holds independently of $N\in\mathbb{N}$, and independently of the sequences $\{a_\nu\}_{\nu=1}^N$ and $\{\lambda_\nu\}_{\nu=1}^N$.

For $h_1(t)=(it)^{-1}$ the problem of finding the best possible values for $L(h_1)$ and $U(h_1)$ was solved by Montgomery and Vaughan [15]. As mentioned in the introduction, Beurling's majorant $B(z)$ can be used to give a proof of Montgomery and Vaughan's result (cf. [25] Theorem 16, [22] chapter 20). We extend their result to the functions

$$h_m(t)=(it)^{-m} \text{ where } m\in\mathbb{N}. \tag{4.22}$$

Corollary 4.3.1. *Let $m\in\mathbb{N}$, let $\delta>0$, and let L_δ, U_δ be as in (4.21). We have the optimal bounds*

$$\begin{aligned} L_\delta\big((it)^{-m}\big) &= (2\pi)^m\frac{B_m(\alpha_{m-1})}{m!\delta^m},\\ U_\delta\big((it)^{-m}\big) &= -(2\pi)^m\frac{B_m(\beta_{m-1})}{m!\delta^m}. \end{aligned}$$

Proof. We prove the corollary for $\delta=1$. The general case follows by noting that $|\lambda_\nu-\lambda_\mu|\ge\delta>0$ implies $|\lambda_\nu/\delta-\lambda_\mu/\delta|\ge 1$.

From Corollary 4.2.1 (i) we obtain that for any $N\in\mathbb{N}$, $a_\nu\in\mathbb{C}$, and $\lambda_\nu\in\mathbb{R}$

$$\sum_{\nu,\mu=1}^N a_\nu\overline{a}_\mu f_m(\lambda_\nu-\lambda_\mu)\ge 0.$$

If we require additionally that $|\lambda_\nu-\lambda_\mu|\ge 1$ for all $\nu\neq\mu$, then after a multiplication by $m!^{-1}(2\pi)^m$ we obtain

$$\sum_{\substack{\nu,\mu=1\\ \nu\neq\mu}}^N a_\nu\overline{a}_\mu(i(\lambda_\nu-\lambda_\mu))^{-m}\ge -f_m(0)\frac{(2\pi)^m}{m!}\sum_{\nu=1}^N|a_\nu|^2.$$

This shows that the function $(it)^{-m}$ satisfies (4.21) with $L_1((it)^{-m})$ as in Corollary 4.3.1. The optimality of $L_1((it)^{-m})$ follows from (4.20). (Note that the set of integers $\{\lambda_\nu\}$ used in (4.20) obviously satisfies $|\lambda_\nu - \lambda_\mu| \geq 1$ for all $\nu \neq \mu$.)

The validity of $U_1((it)^{-m})$ is verified in the same way using Corollary 4.2.1, (ii) and (iv). $\square$

For example, since $-2\pi^2 B_2(1/2) = \pi^2 B_2(0) = \zeta(2)$ we obtain for $m = 2$ and $\delta = 1$ that

$$-\zeta(2)\sum_{\nu=1}^{N}|a_\nu|^2 \leq \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{N}\frac{a_\nu\overline{a}_\mu}{(\lambda_\nu-\lambda_\mu)^2} \leq 2\zeta(2)\sum_{\nu=1}^{N}|a_\nu|^2 \tag{4.23}$$

for all $N \in \mathbb{N}$ and all sequences (a_ν), $\{\lambda_\nu\}$ as above.

For $m = 2$ and $\delta = 1$ we have the following extremal configurations. An extremal configuration for the upper bound is given by $\lambda_\nu := \nu$, $a_\nu := 1$, and $N \to \infty$, since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{\nu,\mu=1\\ n\neq m}}^{N}\frac{1}{(\nu-\mu)^2} = \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{k=1-N\\ k\neq 0}}^{N-1}\frac{N-|k|}{k^2} = 2\zeta(2).$$

An extremal configuration for the lower bound is given by $\lambda_\nu := \nu$, $a_\nu := (-1)^\nu$ and $N \to \infty$, since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{\nu,\mu=1\\ n\neq m}}^{N}\frac{(-1)^{\nu-\mu}}{(\nu-\mu)^2} = \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{k=1-N\\ k\neq 0}}^{N-1}(-1)^k\frac{N-|k|}{k^2} = -\zeta(2).$$

Note that $L_\delta((it)^{-m}) = U_\delta((it)^{-m})$ for odd-valued, but not for even-valued $m \in \mathbb{N}$.

Recall that $|\lambda_\mu - \lambda_\nu| \geq \delta > 0$ for all $\mu \neq \nu$. Letting $n = 0$ in Corollary 4.3.1, we see that (4.21) is equivalent to the inequality

$$\Big|\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{N}\frac{a_\nu\overline{a}_\mu}{\lambda_\nu-\lambda_\nu}\Big| \leq \pi\delta^{-1}\sum_{\nu=1}^{N}|a_\nu|^2. \tag{4.24}$$

For $f \in L^2(\mathbb{R})$ define the Hilbert transform of f by

$$Hf(x) := \int_{-\infty}^{\infty}\frac{f(t)}{t-x}dt, \tag{4.25}$$

where the integral is understood to be the Cauchy principal value. It is well known that H satisfies the inequality

$$\Big|\int_{-\infty}^{\infty} Hf(t)\overline{f(t)}dt\Big| \leq \pi \int_{\infty}^{\infty} |f(t)|^2 dt \tag{4.26}$$

for every $f \in L^2(\mathbb{R})$. We remark that (4.24) implies (4.26) for Riemann-integrable functions f by taking $a_\nu := f(\lambda_\nu)$, since after division by δ, (4.24) is a Riemann sum of (4.26).

4.4 Geometric Consequences

As another application we derive the following result originally obtained by J. J. Holt (cf. [7], Theorem 1 and Corollary 1). Let $\alpha > 0$, and define

$$R_\alpha(x) := \alpha^{-1}(|x+\alpha| - |x|) \text{ for all } x \in \mathbb{R}. \tag{4.27}$$

Holt obtained extremal majorants and minorants of type 2π for $R_\alpha(x)$ in the case that $\alpha \in A := (0, 1/2] \cup \{k + 1/2 : k \in \mathbb{N}\}$, and he obtained non-extremal minorants and majorants of type 2π for all other $\alpha > 0$. We obtain Holt's result for $\alpha \in A$, and we give slightly better (also non-extremal) majorants and minorants for all positive $\alpha \notin A$. Define

$$M_\alpha(x) = \begin{cases} H_0(x;0) & \text{if } 0 < \alpha \leq 1/2 \\ \alpha^{-1}(H_1(x+\alpha;1/2) - H_1(x;0)) & \text{if } \alpha > 1/2. \end{cases} \tag{4.28}$$

$$m_\alpha(x) = \begin{cases} H_0(x+\alpha;1) & \text{if } 0 < \alpha \leq 1/2 \\ \alpha^{-1}(H_1(x+\alpha;0) - H_1(x;1/2)) & \text{if } \alpha > 1/2. \end{cases} \tag{4.29}$$

For $0 < \alpha \leq 1/2$ we have $H_0(x+\alpha;1) \leq R_\alpha(x) \leq H_0(x;0)$ (cf. [7], Cor. 1). For any $\alpha > 0$ we have by Theorem 2.5.3 that $H_1(x+\alpha;0) \leq |x+\alpha| \leq H_1(x+\alpha;1/2)$ and $-H_1(x;1/2) \leq -|x| \leq -H_1(x;0)$. So for all $x \in \mathbb{R}$

$$m_\alpha(x) \leq R_\alpha(x) \leq M_\alpha(x). \tag{4.30}$$

Moreover for $0 < \alpha \leq 1/2$, $\int(H_0(x;0) - R_\alpha(x))dx = \int(R_\alpha(x) - H_0(x+$

$\alpha; 1))dx = 1 - \alpha$. Since $-B_2(1/2) + B_2(0) = 1/12 + 1/6 = 1/4$, Theorem 2.5.3 implies for $\alpha > 1/2$

$$\int_{\mathbb{R}} (M_\alpha - R_\alpha) = \int_{\mathbb{R}} (R_\alpha - m_\alpha) = (4\alpha)^{-1}. \tag{4.31}$$

Define

$$d(\alpha) = \begin{cases} 1 - \alpha & \text{if } 0 < \alpha \leq 1/2 \\ (4\alpha)^{-1} & \text{if } \alpha > 1/2. \end{cases} \tag{4.32}$$

We have shown

Corollary 4.4.1. *The functions M_α and m_α are of type 2π, and they majorize and minorize R_α, respectively, on the real line. Moreover,*

$$\int_{\mathbb{R}} (M_\alpha - R_\alpha) = \int_{\mathbb{R}} (R_\alpha - m_\alpha) = d(\alpha).$$

We use Corollary 4.4.1 to obtain majorants and minorants of type 2π for trapezoids. Define $f_{\alpha,\beta,\gamma}(x) = \frac{1}{2}(R_\alpha(x) + R_\gamma(\beta - x))$. The graph of $f_{\alpha,\beta,\gamma}(x)$ is a trapezoid with base-length $\alpha + \beta + \gamma$, top-length β, height 1, and left point at $x = -\alpha$. Define

$$M_{\alpha,\beta,\gamma}(x) = \frac{1}{2}(M_\alpha(x) + M_\gamma(\beta - x)), \tag{4.33}$$

$$m_{\alpha,\beta,\gamma}(x) = \frac{1}{2}(m_\alpha(x) + m_\gamma(\beta - x)). \tag{4.34}$$

From Corollary 4.4.1 we obtain

Corollary 4.4.2. *$M_{\alpha,\beta,\gamma}$ and $m_{\alpha,\beta,\gamma}$ are functions of type 2π, they satisfy*

$$m_{\alpha,\beta,\gamma}(x) \leq f_{\alpha,\beta,\gamma}(x) \leq M_{\alpha,\beta,\gamma}(x)$$

for all real x, and

$$\int_{\mathbb{R}} (M_{\alpha,\beta,\gamma} - f_{\alpha,\beta,\gamma}) = \int_{\mathbb{R}} (f_{\alpha,\beta,\gamma} - m_{\alpha,\beta,\gamma}) = \frac{1}{2}(d(\alpha) + d(\gamma)).$$

References

[1] M. Abramowitz and I. A. Stegun (ed.), *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, June 1964.

[2] A. Beurling, *On functions with a spectral gap*, Seminar on Harmonic Analysis, Univ. of Uppsala. March 24, 1942. Published in *Collected Works of Arne Beurling: Vol. II, Harmonic Analysis*, Birkhäuser, Boston, 1989, 359–364.

[3] R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.

[4] H. B. Curry and I. J. Schoenberg, *On Pólya frequency functions. IV. The fundamental spline functions and their limits*, J. Analyse Math. **17**, (1966), 71–107.

[5] S. W. Graham and J. D. Vaaler, *A class of extremal functions for the Fourier transform*, Trans. Amer. Math. Soc. **265** (1981), 283–302.

[6] I. I. Hirschman and D. V. Widder, *The Convolution Transform*. Princeton University Press 1955, Princeton, New Jersey.

[7] J. J. Holt, *A class of extremal functions and trigonometric polynomials*, J. Approx. Theory **88** (1997), 275–303.

[8] J. J. Holt and J. D. Vaaler, *The Beurling-Selberg extremal functions for a ball in Euclidean space*, Duke Math. J. **83** (April 1996), No. 1, 203–247.

[9] S. Karlin, *Total Positivity I*, Stanford University Press, Stanford, California, 1968.

[10] J. Korevaar, *A century of complex Tauberian theory*, Bull. Amer. Math. Soc. **39** (October 2002), 475–532.

[11] E. Laguerre, *Sur les fonctions du genre zéro et du genre un*, Comptes Rendus des Séances de l'Académie des Sciences **98**, 828–831.

[12] D. H. Lehmer, *On the maxima and minima of Bernoulli polynomials*, Amer. Math. Monthly **47** (1940), 533–538.

[13] L. M. Milne-Thomson, *The Calculus of Finite Differences*, MacMillan and Co., London 1933.

[14] H. Montgomery, *The analytic principle of the large sieve*, Bull. Amer. Math. Soc. **84** (July 1978), 547–567.

[15] H. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. **8** (1974), 73–81.

[16] N. E. Nörlund, *Vorlesungen über Differenzenrechung*, Chelsea Publishing Company, New York 1954.

[17] M. Plancherel and G. Pólya, *Fonctions entières et intégrales de Fourier multiples*, (seconde partie) Comment. Math. Helv. **10**, (1938), 110–163.

[18] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, International Editions, 1987.

[19] I. J. Schoenberg, *On Pólya Frequency Functions. I. The totally positive functions and their Laplace transforms*. J. Analyse Math. **1**, (1951), 331–374, (I. J. Schoenberg, *Selected Papers 2*, Birkhäuser Verlag 1988.)

[20] I. J. Schoenberg, *On Pólya frequency functions. II. Variation-diminishing integral operators of the convolution type*, Acta Sci. Math. (Szeged) **12**, (1950), 97–106.

[21] I. J. Schoenberg and A. Whitney, *On Pólya frequency functions. III. The positivity of translation determinants with an application to the interpolation problem by spline curves*, Trans. Amer. Math. Soc. **74**, (1953), 246–259.

[22] A. Selberg, *Lectures on sieves*, Collected Papers II, Springer-Verlag, New York, 1991.

[23] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton 1990.

[24] N. M. Temme, *Special Functions*, John Wiley & Sons, Inc., New York, 1996.

[25] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. **12** (April 1985), 183–216.

[26] A. H. Zemanian, *Distribution Theory and Transform Analysis*, Dover Publications Inc., New York 1987.

Vita

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