# Class 1/19

## 1 Möbius transformations

We define $\mathbb{C}_\infty$ to be the complex plane with $\infty$ (this is a compact space that can be visualized with aid of the stereographic projection). We note that $1/\infty = 0$ and $1/0 = \infty$.

**Definition 1.** A Möbius transformation is a linear transformation

$$S(z) = \frac{az+b}{cz+d} \tag{1}$$

with $a, b, c, d \in \mathbb{C}$ so that $ad - bc \neq 0$.

The inverse map of (1) is

$$S^{-1}(z) = \frac{dz-b}{-cz+a}$$

and compositions of Möbius transformations are Möbius transformations (the coefficients of the composition can be obtained by a matrix multiplication, which implies in particular that the non-zero condition is satisfied). The identity acts as the neutral element, hence Möbius transformations form a (non-commutative) group.

**Proposition 1.** *Every Möbius transformation is generated by compositions of translations $z \mapsto z+u$ with $u \in \mathbb{C}$, dilations $z \mapsto rz$ with $r > 0$, rotations $z \mapsto e^{i\theta}z$ with real $\theta$, and inversion $z \mapsto z^{-1}$.*

*Proof.* This is clear if $c = 0$ in (1). For $c \neq 0$ it follows from

$$\frac{(bc-ad)}{c^2}\left[\frac{1}{z+\frac{d}{c}}\right] + \frac{a}{c} = \frac{az+b}{cz+d}$$

and the fact that the complex constant at the beginning of the left side is non-zero and hence can be written as $re^{i\theta}$ with $r > 0$ and real $\theta$. □

**Proposition 2.** *A Möbius transformation has $\leq 2$ or infinitely many fixed points.*

*Proof.* Multiplication of $\frac{az+b}{cz+d} = z$ with $cz + d$ leads to a quadratic equation. The coefficients of the quadratic are zero if and only if the transformation is the identity. □

In particular, any Möbius transformation is determined by the images of three distinct points. (If there were two different transformations $S, T$ having identical images at three distinct points, then $S \circ T^{-1}$ would have three fixed points and therefore be the identity map.)

Notation: We write $S(z) = (z, z_2, z_3, z_4)$ for the Möbius transformation that satisfies

$$S(z_2) = 1, S(z_3) = 0, S(z_4) = \infty.$$

If $z_1, z_2, z_3 \in \mathbb{C}$, then

$$S(z) = \frac{z_2 - z_4}{z_2 - z_3} \cdot \frac{z - z_3}{z - z_4}.$$

(Note that the second fraction maps $z_3$ and $z_4$ to the correct values, and the first fraction is chosen so that $z_2$ is mapped to 1.) If $z_2 = \infty$ then $S(z) = \frac{z-z_3}{z-z_4}$, if $z_3 = \infty$ then $S(z) = \frac{z_2-z4}{z-z_4}$, if $z_4 = \infty$ then $S(z) = \frac{z-z_3}{z_2-z_3}$.

**Definition 2.** A cricle in $C_\infty$ is the set of points that satisfies

$$\alpha z\overline{z} + \beta z + \overline{\beta z} + \gamma = 0 \tag{2}$$

where $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$, and $|\beta|^2 > \alpha\gamma$.

This is either a circle in $\mathbb{C}$ ($\alpha \neq 0$) or a straight line ($\alpha = 0$). To see this, rewrite the equation in terms of $x$ and $y$: if $\beta = b_1 + ib_2$, then we obtain

$$\alpha(x^2 + y^2) + 2(b_1 x - b_2 y) + \gamma = 0,$$

now complete the squares.

**Theorem 1.** *A Möbius transformation maps circles in $C_\infty$ to circles in $C_\infty$.*

*Proof.* We need to prove that points satisfying an equation of the form (2) are mapped to a set that can be described by another equation of this form. It suffices to do this for translations, rotations, dilations and inversion, since they generate all Möbius transformations under composition.

This is a direct check; for example if $S(z) = z^{-1}$, then set $w = z^{-1}$ and assume that $z$ satisfies eqrefcircle-eq. Substitute and multiply by $|w|^2$ to obtain

$$\alpha + \beta\overline{w} + \overline{\beta}w + \gamma w\overline{w} = 0,$$

which is also of the form (2). □

Consequence: every circle can be mapped onto every line with a Möbius transformation. In particular, Möbius transformations can be used to map disks to half planes with analytic, even conformal, maps.