Math 752 **Spring 2011**

Stereographic projection

The following is an analytic proof of the fact that the stereographic projection sends circles and lines in $\mathbb{C}$ to circles on the sphere. (There are geometric ones as well.)

The stereographic projection of a point in (u, v)-space has coordinates

$$(x, y, z) = \Big(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\Big).$$

A circle on the sphere is an intersection of the sphere with a plane

$$ax + by + cz = d. \tag{1}$$

Substitute the stereographic projection into the plane equation and simplify. This leads (eventually) to the identity

$$(c-d)(u^2+v^2) + 2au + 2bv - (c+d) = 0. \tag{2}$$

If $c = d$ then (2) describes a line in (u, v)-space. On the other hand, $c = d$ in the plane equation (1) means that the point $(0, 0, 1)$ lies in the plane. This is the north pole of the sphere, so circles through the north pole on the sphere map to lines in (u, v)-space. If $c \neq d$, then (2) describes a circle in (u, v)-space. This calculation can be reversed; if we start with a point satisfying (2), then its stereographic projection satisfies $ax + by + cz = d$ (and lies on the sphere).