

**DIFFERENTIAL EQUATIONS MATH 266 SPRING 2014 (COHEN) LECTURE
NOTES**

1 Functions, Derivatives, and Notation

A **function**, informally, is an input-output correspondence between values in a **domain** or input space, and values in a **codomain** or output space. When studying a particular function in this course, our domain will always be either a subset of the set of all real numbers, always denoted \mathbb{R} , or a subset of n -dimensional Euclidean space, always denoted \mathbb{R}^n . Formally, \mathbb{R}^n is the set of all ordered n -tuples (x_1, x_2, \dots, x_n) where each of x_1, x_2, \dots, x_n is a real number. Our codomain will always be \mathbb{R} , i.e. our functions under consideration will be **real-valued**.

We will typically functions in two ways. If we wish to highlight the **variable** or **input parameter**, we will write a function using first a letter which represents the function, followed by one or more letters in parentheses which represent input variables. For example, we write

$$f(x) \text{ or } y(t) \text{ or } F(x, y).$$

In the first two examples above, we implicitly assume that the variables x and t vary over all possible inputs in the domain \mathbb{R} , while in the third example we assume (x, y) varies over all possible inputs in the domain \mathbb{R}^2 . For some specific examples of functions, e.g. $f(x) = \sqrt{x-5}$, we assume that the domain includes only real numbers x for which it makes sense to plug x into the formula, i.e., the domain of $f(x) = \sqrt{x-5}$ is just the interval $[5, \infty)$.

Whenever it does not lead to confusion, we will refer to functions just by their names f , y , F , etc., and omit reference to the input variable. This shorthand can occasionally be cause for confusion, and so the student will be asked to parse which letters are input variables and which are functions from the context. For example, in the differential equation

$$y' = \frac{x^2}{y^2} \cos y,$$

the use of the single-variable derivative notation y' implies that $y = y(x)$ is a function of the single input variable x . On the other hand in the differential equation

$$f_{xx} + f_{yy} = 0,$$

we note the use of partial derivative notation, and assume that $f = f(x, y)$ is a function of the two variables x and y . (More comments on derivative notation below.)

2 Remarks on Leibniz Notation (For the Students' Reading Pleasure - Not To Be Covered in Lecture)

The student will have seen at least two common families of notation for derivatives and partial derivatives. One kind looks like:

$$y', f''(x), y^{(4)}, f_x, f_{xy}(x, y), \dots, \text{etc.},$$

while the other looks like

$$\frac{dy}{dx}, \frac{d^2}{dx^2} f(x), \frac{d^4 y}{dt^4}, \frac{\delta f}{\delta x}, \frac{\delta^2}{\delta y \delta x} f(x, y), \text{etc.}$$

The former group is often referred to as the **Lagrange notation**, while the latter group is the **Leibniz notation**. Both are acceptable and commonly used in practice, and have distinct advantages and disadvantages over one another. The major advantage of the Leibniz notation is that it is completely explicit about which variable the given derivative respects. However, the Leibniz notation is notoriously a source of great confusion for calculus students (this fact may be mostly blamed on obsolete textbook authorship), and so I ask the student to thoroughly understand its meaning before we proceed to the main content of the course.

The major drawback of the notation $\frac{dy}{dx}$ is that it is misleading: it looks like a fraction. *The derivative $\frac{dy}{dx}$ is in no way a fraction:* if $y = y(x)$ is a real-valued function of a single-variable, then the derivative $\frac{dy}{dx} = \frac{dy}{dx}(x)$ is a real-valued function of a single variable.

Moreover, *the components of the notation, dy and dx , are not rigorously defined mathematical objects of their own.* Many textbooks refer to dy , dx , and similar symbols as “infinitesimals,” “differentials” or “elements” (Isaac Newton called them “fluxions”!). This is an outdated misuse of terminology that should be abolished from the student’s brain and vocabulary. The notation $\frac{dy}{dx}$ is to be regarded as a single unified notation which cannot be separated into constituent parts. For instance, we may write $\frac{dy}{dx} = 5x$, which is a meaningful mathematical statement, but we do not consider it meaningful to “multiply on both sides by dx ” to obtain the statement “ $dy = 5x dx$.” Unfortunately many authors choose to use expressions like $dy = 5x dx$ on a regular basis. Whenever we see a statement of the latter form, we will regard it mentally as just a shorthand for a statement of the former form.

Why has the “fraction notation” become so entrenched when we are not really working with fractions at all? One popular feature of the Leibniz notation is that it synergizes well with the chain rule for derivatives and the substitution rule for integrals, in the sense that it makes them look nice visually. Recall that if $y(x)$ and $u(x)$ are real-valued functions of a single variable x , then the composition $y(u(x))$ is a real-valued function of x , and the chain rule says that its derivative is equal to

$$\frac{d}{dx}y(u(x)) = y'(u(x)) \cdot u'(x).$$

People often switch to the Leibniz notation to write the chain rule in the following condensed form:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

This form is aesthetically pleasing because it looks like cancellation of fractions (again, a source of confusion). But there is clearly no cancellation happening; the above statement is of the form “function equals function times function.” Moreover, the statement above is ambiguous because the notation $\frac{dy}{du}$, where y and u are both functions, is rarely if ever defined carefully at the introductory calculus level, and the input variables, which are of crucial importance, are omitted. The useful content of the chain rule is that the derivative of the composition $y \circ u$ may be found by computing $\frac{dy}{du}$ and $\frac{du}{dx}$ separately, and then adjusting the input variables. In other words, a correct version of the chain rule, written in Leibniz notation, should look like:

$$\frac{dy}{dx}(x) = \frac{dy}{du}(u(x)) \cdot \frac{du}{dx}(x),$$

which helps shatter any illusion of fraction cancellation.

A similar problem is encountered when we work with the substitution rule for integrals:

$$\int_a^b y'(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} y'(u) du.$$

(Crucially recall here that the string of symbols $\int_a^b \text{-----} dx$ means “take the integral of ----- over the interval $[a, b]$ with respect to x ”; the dx is again an inseparable part of the notation and should not be regarded as a mathematical object all on its own.)

The substitution rule is usually applied in practice by means of the following mnemonic device: “Set $u = g(x)$. Then $\frac{du}{dx} = g'(x)$, and u runs over $[g(a), g(b)]$ as x runs over $[a, b]$, so

$$\int_a^b y'(g(x)) \cdot g'(x) dx = \int_a^b y'(u) \frac{du}{dx} dx = \int_{g(a)}^{g(b)} y'(u) du.”$$

Again we obtain the illusion of fraction cancellation, caused by the judicious omission of input variables, together with a highly suspect duplication of the symbol u as both a function and a dummy variable of integration.

The moral of the story here is that the Leibniz notation can be useful (particularly as a mnemonic device for certain calculus techniques) but it comes with many misleading characteristics, and so I will ask the student to be very careful mentally when reading and working with these symbols. Especially be aware, please, when an author is using “differentials” (dx, dy, dt etc.) as a mnemonic device or as a non-rigorous shorthand notation.

As a last remark, the rigorousness of infinitesimals and infinitesimal notation was a major debate for many decades and has become an interesting and important part of mathematical history. Our modern approach to calculus avoids reference to infinitesimals by instead relying on the notion of a **limit**; this approach is essentially due to Karl Weierstrass. It is actually possible to make things like *differentials* and *infinitesimals* dy, dx , etc. rigorous notions (for example, google: “differential geometry” or “non-standard analysis”) but these tools are well beyond the scope of our course and don’t really shed any light on our present purposes. So we will regard dx, dy , etc. purely as notational conveniences, at their best, and confusing encumbrances at their worst.

3 Terminology of Differential Equations

Definition 3.1. A **differential equation** is an equation relating some function to its derivatives. An **ordinary differential equation (ODE)** is a differential equation involving a function of a single-variable. A **partial differential equation (PDE)** is a differential equation involving a function of multiple variables. A **solution** to a differential equation is any function which satisfies the equation.

Example 3.2 (Examples of Differential Equations). .

ODE's	PDE's
$\frac{dv}{dt} = -32$	$u_{xx} + u_{yy} = 0$
$y' = 3x^2 - 4x$	$f_x + f f_{yy} = 0$
$y'' + y = 0$	$\frac{\delta u}{\delta x} \frac{\delta u}{\delta y} = u$
$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$	$F_x = 5x^2y - 3y$
$y'' + y' - 2y = x^3$	

Example 3.3. (1) Find all solutions to the differential equation $y' = 3x^2 - 4x$.

(2) Find at least one solution to the differential equation $y'' + y = 0$.

Example 3.4. Find the unique solution to the differential equation $\frac{dv}{dt} = -32$ which satisfies the **initial condition** $v(0) = -64$.

Definition 3.5 (Informal). An **initial value problem (IVP)** is a differential equation together with one or more initial conditions.

Example 3.6. Find the solution to the IVP $y' = 3x^2 - 4x, y(1) = 4$.

Definition 3.7. The **order** of a differential equation is the highest degree of derivative which appears in the equation.

Example 3.8. Find the order of each of the following differential equations.

(1) $\frac{dy}{dx} = \frac{x^2}{y^2} \cos y$ (degree 1)

$$(2) \quad u_{xx} + u_{yy} = 0 \quad (\text{degree } 2)$$

$$(3) \quad F_{xx} - F_{xyx} = xy \quad (\text{degree } 3)$$

$$(4) \quad \left(\frac{dy}{dx}\right)^4 = y + x \quad (\text{degree } 1)$$

$$(5) \quad \frac{d^7x}{dt^7} + 2\frac{dx}{dt} + 3x = \sin t \quad (\text{degree } 7)$$

Example 3.9. (1) Show that $y = c_1 \sin t + c_2 \cos t$ satisfies the ODE $y'' + y = 0$ for all real numbers c_1, c_2 .

(2) Find a solution to the IVP $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$.

Definition 3.10. An ODE (involving the function y of the independent variable x) of order n is called **linear** if it may be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x),$$

where a_0, a_1, \dots, a_n and f are functions of x and a_n is not the zero function. If f is identically the 0 function, then the ODE is called **homogeneous**. Note that the function $y(x) = 0$ is a solution to any homogeneous ODE, called the **trivial solution**.

Example 3.11. Determine if the following ODE's are (a) linear or nonlinear, and (b) homogeneous or nonhomogeneous.

$$(1) \quad \frac{dy}{dx} + x^2y = x \quad (\text{linear nonhomogeneous})$$

$$(2) \quad \frac{dy}{dx} = x^3 \quad (\text{linear nonhomogeneous})$$

$$(3) \quad y - 1 + (x \cos y)y' = 0 \quad (\text{nonlinear})$$

$$(4) \quad u'' + u - e^x = 0 \quad (\text{linear nonhomogeneous})$$

$$(5) \quad \frac{d^2f}{dx^2} = -f \quad (\text{homogeneous})$$

$$(6) \quad y^{(3)} + yy' = x \quad (\text{nonlinear})$$

$$(7) \quad \frac{d^2x}{dt^2} + \sin x = 0 \quad (\text{nonlinear})$$

Example 3.12. Solve the following differential equations.

$$(1) \quad y' = \cos x \quad (\text{easy})$$

$$(2) \quad y' = \frac{x}{\sqrt{x^2 + 1}} \quad (\text{substitution rule})$$

$$(3) \quad y' = \frac{1}{x^2 + 16} \quad (\text{trig substitution})$$

$$(4) \quad y' = xe^x \quad (\text{integration by parts})$$

$$(5) \quad y' = \frac{1}{4 - x^2} \quad (\text{partial fractions})$$

$$(6) \quad y' = y \quad (\text{remembering what we already know + cleverness})$$

Definition 3.13 (Informal). Given an ODE (involving a function y of an independent variable x), an *explicit* solution is a function $y = y(x)$ which satisfies the equation. An *implicit solution* is an equation in y and x , such that if a given differentiable function satisfies the equation, then the function also satisfies the given ODE.

Example 3.14. Show that an implicit solution to the ODE

$$y' = \frac{2y - 4x - 5}{2y - 2x}$$

is given by the equation $2x^2 + y^2 - 2xy + 5x = 0$. (Hint: implicit differentiation on the solution.)

Example 3.15 (Visualizing Solutions to ODEs). Sketch the slope field associated to the differential equation $y' = e^{-x^2}$. Sketch a possible solution. (For context, recall that e^{-x^2} does not have a nice explicit antiderivative.)

4 Existence of Solutions to First-Order Differential Equations

Theorem 4.1 (Picard-Lindelöf Existence and Uniqueness Theorem for First-Order ODE's). *Let f be a function in two variables and consider the IVP*

$$y' = f(t, y), \quad y(t_0) = y_0.$$

If f has first-order partial derivatives which are continuous on an open set containing (t_0, y_0) , then there exists an open interval $(t_0 - h, t_0 + h)$ on which there exists one and only one function which is a solution to the IVP.

Example 4.2. Verify that the IVP $y' = y$, $y(0) = 1$ has a unique solution.

5 Separable Equations

Definition 5.1. A first-order ODE is called **separable** if it may be written in the form

$$g(y)y' = f(t),$$

for some functions g, f in one variable. (Note: some authors will use the shorthand $g(y)dy = f(t)dt$; see Section 2 of these notes.)

Example 5.2. We wish to model the population P of a group of apes in the forest, as a function of time t measured in years. Suppose that the birth rate of the apes is proportional to the population, so that the birth rate may be written $bP(t)$ for some constant b . Likewise suppose the mortality rate is proportional to the population, say $mP(t)$ for some constant m . Setting $k = b - m$, then the rate of change of the ape population is modeled by

$$\frac{dP}{dt} = (\text{Birth Rate}) - (\text{Mortality Rate}) = bP - mP = kP.$$

Assuming that the initial population of apes is P_0 , find a model $P(t)$ for the population by solving the above first-order ODE.

Solution. First note that the equation is separable, as the equation $P' = kP$ may be rewritten:

$$\frac{1}{P} \cdot P' = k.$$

Integrate both sides with respect to t :

$$\int \frac{1}{P(t)} P'(t) dt = \int k dt.$$

Note that the left-hand side is a substitution rule problem (mentally set $u = P(t)$). Integration gives us

$$\ln |P(t)| + C_1 = kt + C_2$$

where C_1 and C_2 are arbitrary constants. Before finishing the problem, let us make a simple observation that will serve us the remainder of the course: there is no reason to write two arbitrary constants when one will do. Set $C_3 = C_2 - C_1$ (so C_3 is still arbitrary) and we get

$$\ln |P(t)| = kt + C_3.$$

We finish the problem by exponentiating both sides:

$$|P(t)| = e^{\ln |P(t)|} = e^{kt+C_3} = e^{kt} \cdot e^{C_3}.$$

Setting $C = e^{C_3}$ (so C is still an arbitrary constant) we find the implicit (general) solution $|P(t)| = Ce^{kt}$. From the context we know P must be positive, so we write $P(t) = Ce^{kt}$. Lastly since $P(0) = y_0$, we get that $C = Ce^{k(0)} = y_0$, so the solution to our IVP is

$$P(t) = y_0 e^{kt}.$$

The student should verify that this is indeed a solution to the IVP. □

Technique 5.3 (General Method for Solving Separable First-Order ODE's). (1) First separate all the y -terms onto the one side of the equation and all of the t -terms onto the other side, so the equation is in the form

$$g(y)y' = f(t).$$

(If you wish, you may write $y' = \frac{dy}{dt}$ and use the mnemonic device $g(y)dy = f(t)dt$.)

- (2) Integrate both sides of the equation with respect to t . The left-hand side is a substitution-rule integration, so the y' -term ends up disappearing.
- (3) Solve the equation for y if possible to obtain a general explicit solution. (You may be stuck with just an implicit solution.) Replace arbitrary constants as necessary to keep things simple.

Example 5.4. Solve $y' = -\frac{t}{y}$.

Example 5.5. Solve the IVP $y' = e^{2t+y}$, $y(0) = 0$.

Example 5.6. Solve the IVP $y' = \frac{e^t}{y+1}$, $y(0) = -2$.

Example 5.7. Solve the ODE $y' = \sin(t-y) + \sin(t+y)$.

Definition 5.8. An **autonomous** first-order ODE is one of the form $y' = f(y)$, i.e., an autonomous first-order ODE is one where the independent variable t does not explicitly appear. An **equilibrium solution** to an autonomous ODE is a constant solution, i.e. a function of the form $y(t) = C$ which satisfies the equation.

Example 5.9. Consider $y' = 2y - y^2$.

- (1) Observe that the ODE is autonomous.
- (2) Find a general solution to the ODE.
- (3) Find any equilibrium solutions to the ODE.

6 First-Order Linear Equations

Definition 6.1. Recall that a **first-order linear ODE** is one of the form

$$y' + p(t)y = q(t)$$

for some functions p and q . The **corresponding homogeneous equation** is the ODE:

$$y' + p(t)y = 0,$$

i.e. we obtain the corresponding homogeneous equation by just dropping the $q(t)$ term from the original equation.

Recall from a previous course that if F_1 and F_2 are two functions for which $F_1' = y$ and $F_2' = y$, then $F_2 = F_1 + C$ for some constant C . In other words, any two antiderivatives of a given function y differ only by a constant. By analogy, the next theorem says that any two solutions to a first-order linear ODE differ only by a solution to the corresponding homogeneous equation.

Theorem 6.2. Let $y' + p(t)y = q(t)$ be a first-order linear ODE, and suppose y_1 and y_2 are two particular solutions. Then

$$y_2 = y_1 + y_h,$$

where y_h is some solution to the corresponding homogeneous equation.

Proof. Define the function y_h by setting $y_h = y_2 - y_1$. Then clearly, $y_2 = y_1 + y_h$, so we need only check that y_h is a solution to the corresponding homogeneous equation, given by

$$y' + p(t)y = 0.$$

This is immediate because y_1 and y_2 are solutions to the original ODE; check that:

$$y_h' + p(t)y_h = (y_2' - y_1') + p(t)(y_2 - y_1) = (y_2' + p(t)y_2) - (y_1' + p(t)y_1) = q(t) - q(t) = 0.$$

□

Corollary 6.3 (General Solution for First-Order Linear ODEs, Part I). Let y_p be any particular solution to the first-order linear ODE $y' + p(t)y = q(t)$. Then every solution to the ODE is of the form

$$y_p + y_h,$$

where y_h is some solution to the corresponding homogeneous equation.

Example 6.4. Consider the ODE

$$\frac{dy}{dt} + \frac{3t}{1+t^2}y = \frac{t}{(1+t^2)^2}.$$

- (1) Show that $y_p = \frac{1}{1+t^2}$ is a particular solution.
- (2) Find all solutions to the ODE.

Solution. Part (a) is a straightforward computation, so we take it as a fact and move on to part (b). By the previous corollary, to find a general solution to the ODE it suffices to find a general solution to the corresponding homogeneous equation

$$y' + \frac{3t}{1+t^2}y = 0.$$

Thankfully this latter equation is separable:

$$\left(\frac{1}{y}\right) y' = -\frac{3t}{1+t^2}.$$

Integrating both sides (substitution rule on both sides, with $u = 1 + t^2$ on the right-hand side) yields

$$\ln |y| = -\frac{3}{2} \ln |1 + t^2| + C.$$

Exponentiate both sides (and replace the arbitrary constant):

$$|y| = e^C e^{-(3/2) \ln |1+t^2|} = C |1 + t^2|^{-3/2}.$$

Noting that $(1 + t^2)$ is always positive, and that the positivity or negativity of y may be absorbed into the arbitrary constant C , we drop absolute value signs and arrive at the general solution

$$y = C(1 + t^2)^{-3/2}.$$

Since a particular solution to the original ODE is $\frac{1}{1+t^2}$, we conclude that a general solution to the original ODE looks like

$$y = \frac{1}{1+t^2} + C(1+t^2)^{-3/2}.$$

(The student should verify that this solution works.) □

We would like to develop a complete technique for solving first-order linear ODEs. To that end, we introduce the following definition, which is otherwise impossible to motivate.

Definition 6.5. Let $y' + p(t)y = q(t)$ be a first-order linear ODE, where $p(t)$ is continuous. An **integrating factor** $\mu(t)$ associated to the ODE is any function of the form

$$\mu(t) = e^{\int_c^t p(u) du},$$

where c is an arbitrary constant in the domain of p . In other words, μ is the exponentiation of any antiderivative of p .

Theorem 6.6. Let $y' + p(t)y = q(t)$ be a first-order linear ODE, where $p(t)$ is continuous. Then the general solution to the corresponding homogeneous equation $y' + p(t)y = 0$ is

$$y = C\mu(t)^{-1},$$

where μ is any integrating factor and C is any constant. Using the indefinite integral notation, we may write the general solution as

$$y = e^{-\int p(t) dt}.$$

Proof. Use the separability of the corresponding homogeneous equation:

$$\left(\frac{1}{y}\right) y' = -p(t),$$

and integrate both sides to obtain

$$\ln |y| = - \int p(t) dt.$$

Let $g(t)$ be any antiderivative of $p(t)$, and set $\mu(t) = e^{g(t)}$, so μ is an integrating factor. Since any two antiderivatives of $p(t)$ differ only by a constant, we may write

$$\ln |y| = -g(t) + C.$$

Exponentiating both sides, and consolidating arbitrary constants, yields the desired result:

$$y = Ce^{-g(t)} = C\mu(t)^{-1}.$$

□

Technique 6.7 (General Method for Solving First-Order Linear ODEs). (1) Given a first-order linear ODE of the form $y' + p(t)y = q(t)$, choose a representative $\mu(t)$ from the class of integrating factors $e^{-\int p(t) dt}$.

(2) Multiply both sides of the differential equation by $\mu(t)$ to obtain $y'\mu(t) + \mu(t)p(t)y = \mu(t)q(t)$. Note that since $\mu'(t) = \mu(t)p(t)$, this equation may be rewritten as

$$y'(t)\mu(t) + \mu'(t)y(t) = \mu(t)q(t).$$

(3) Integrate both sides with respect to t . By the product rule, the left-hand side has $y(t)\mu(t)$ for an antiderivative.

(4) Solve for y if possible to obtain a general explicit solution.

Corollary 6.8 (General Solution for First-Order Linear ODEs, Part II). *Let $y' + p(t)y = q(t)$ be a first-order linear ODE where p and q are continuous, and let $\mu(t)$ be an integrating factor. Then the general solution to the ODE is given by*

$$y = \mu(t)^{-1} \int q(t)\mu(t) dt.$$

Example 6.9. Solve the IVP $y' + 5t^4y = t^4$, $y(0) = 7$.

Example 6.10. Find the general solution to the ODE $\frac{dr}{dt} = \sin t - r \tan t$, $0 < t < \frac{\pi}{2}$.

Example 6.11. Show that the IVP $ty' - y = t^2 \cos t$, $y(0) = 0$ has infinitely many solutions. Why does this not contradict the Picard-Lindelöf Theorem 4.1?

Example 6.12. Solve $y' - \frac{3t}{(t^2 - 4)}y = t$.

7 Non-Linear Special Case: Exact Equations

Definition 7.1. A first-order ODE of the form

$$M(t, y) + N(t, y)y' = 0$$

is called **exact** if there exists a differentiable function $f(a, b)$ for which

$$M(t, y) = f_a(t, y) \text{ and } N(t, y) = f_b(t, y).$$

Such a function f is called **potential function**.

(To make a connection with multivariable calculus, note that an ODE is exact if and only if the vector field (M, N) is conservative, i.e. $(M, N) = \nabla f$ for some potential function f .)

Theorem 7.2 (Test for Exactness). *Suppose M and N have continuous partial derivatives on an open domain D . The first-order ODE $M(t, y) + N(t, y)y' = 0$ is exact if and only if*

$$\frac{\partial}{\partial y}M = \frac{\partial}{\partial t}N.$$

Example 7.3. Determine whether the following equations are exact.

(1) $2ty^3 + (1 + 3t^2y^2)y' = 0$

(2) $t^2y + 5ty^2y' = 0$

Theorem 7.4 (Multivariable Chain Rule). *Let $f(a, b)$ be a differentiable function of two variables and let $g(t)$ and $h(t)$ be differentiable functions of one variable. Then*

$$\frac{d}{dt}f(g(t), h(t)) = f_a(g(t), h(t))g'(t) + f_b(g(t), h(t))h'(t).$$

Technique 7.5 (General Method for Solving First-Order Exact ODEs). (1) Given an exact equation $M(t, y) + N(t, y)y' = 0$, find a potential function $f(a, b)$ for which $M(t, y) = f_a(t, y)$ and $N(t, y) = f_b(t, y)$, and rewrite the equation as

$$f_a(t, y) \cdot 1 + f_b(t, y) \cdot y' = 0.$$

(2) Note, using the multivariable chain rule, that the above is of the form $\frac{d}{dt}f(t, y(t)) = 0$. Therefore a general solution to the ODE is $f(t, y) = C$ for an arbitrary constant C .

(3) Solve for y if possible to obtain a general explicit solution.

Example 7.6. Find a general solution of $\sin y + y \cos t + (\sin t + t \cos y)y' = 0$.

Example 7.7. Solve the IVP $2t \sin y + (t^2 \cos y - 1)y' = 0$, $y(0) = \frac{1}{2}$.

Example 7.8. Solve $e^{y/t} - \frac{y}{t}e^{y/t} + \frac{1}{1+t^2} + e^{y/t}y' = 0$.

8 Non-Linear Special Case: Bernoulli Equations

Definition 8.1. A **Bernoulli equation** is a first-order ODE of the form

$$y' + p(t)y = q(t)y^n$$

for some integer n .

Technique 8.2 (General Method for Solving Bernoulli Equations). (1) Given an equation of the form $y' + p(t)y = q(t)y^n$, multiply throughout by $(1 - n)y^{-n}$ to obtain

$$(1 - n)y^{-n}y' + (1 - n)p(t)y^{1-n} = (1 - n)q(t).$$

(2) Make the substitution $w = y^{1-n}$, so $w' = (1 - n)y^{-n}y'$. This transforms the equation into

$$w' + (1 - n)p(t)w = (1 - n)q(t).$$

(3) The new equation as above is linear; so solve for w using Technique 6.7.

- (4) Once you have found w , un-substitute and solve for y by taking $y = w^{n-1}$. (Warning: if n is odd, and hence w is an even power of y , then you will probably be stuck with just an implicit solution.)

Example 8.3. Solve $y' + \frac{y}{x} = \frac{1}{xy^2}$, $x > 0$.

Example 8.4. Solve $y' + y = ty^2$.

9 Non-Linear Special Case: Homogeneous Differential Equations

Definition 9.1. Let n be a positive integer. A function $f(a, b)$ of two variables is called **homogeneous of degree n** if

$$f(xa, xb) = x^n f(a, b)$$

for every real number x .

Example 9.2. Show that the function $f(a, b) = 2a^2 - 7b^2 + 4ab$ is homogeneous of degree 2.

Definition 9.3. A first-order ODE is called **homogeneous of degree n** if it can be written in the form

$$M(t, y) + N(t, y)y' = 0$$

where each of M and N is homogeneous of degree n .

Our use of the term *homogeneous* here bears absolutely no relation to its meaning in Definition 3.10. With sympathy for the reader, we remark that *homogeneous* is just one of those words which over time has accumulated an excess of different meanings in mathematical writing (see also for instance *regular* or *normal*), and this is sadly beyond the author's power to change.

Technique 9.4 (General Method for Solving Homogeneous Equations). (1) The strategy here is to make a clever substitution and thereby reduce to the separable case. Given an ODE of the form

$$M(t, y) + N(t, y)y' = 0$$

where both M and N are homogeneous of degree n , choose one of the following substitutions:

$$u = \frac{y}{t} \text{ (hence } y = ut)$$

or

$$v = \frac{t}{y} \text{ (hence } t = vy).$$

As a general guideline, the $y = ut$ substitution tends to be better if $N(t, y)$ is less complicated than $M(t, y)$, whereas the $t = vy$ substitution tends to be better if it's the other way around. If you try one substitution and get stuck, try the other choice.

- (2) If you chose $y = ut$:

- (a) Observe that $y' = u't + u$ by the product rule, and make the substitution to get:

$$M(t, ut) + N(t, ut)(u't + u) = 0.$$

By the homogeneity of M and N , we can rewrite this as:

$$t^n M(1, u) + t^n N(1, u)(u't + u) = 0.$$

- (b) Divide out the
- t^n
- from all terms to obtain

$$M(1, u) + N(1, u)(u't + u) = 0.$$

The resulting equation is separable in t and u :

$$-\left(\frac{N(1, u)}{M(1, u) + uN(1, u)}\right)u' = \frac{1}{t},$$

so proceed as in Technique 5.3 to integrate both sides and obtain an implicit solution in u and t .

- (c) Un-substitute
- $u = \frac{y}{t}$
- , and solve for
- y
- if possible.

- (3) If you chose
- $t = vy$
- :

- (a) Observe that
- $1 = v'y + vy'$
- by the product rule, and make the substitution to get:

$$M(vy, y)(v'y + vy') + N(vy, y)y' = 0.$$

By the homogeneity of M and N , we can rewrite this as:

$$y^n M(v, 1)(v'y + vy') + y^n N(v, 1)y' = 0.$$

- (b) Divide out the
- y^n
- from all terms to obtain

$$M(v, 1)(v'y + vy') + N(v, 1)y' = 0.$$

The equation above is separable in v and y :

$$\left(\frac{1}{y}\right)y' = -\left(\frac{M(v, 1)}{vM(v, 1) + N(v, 1)}\right)v',$$

so proceed as in Technique 5.3 to integrate both sides and obtain an implicit solution in y and v .

- (c) Un-substitute
- $v = \frac{t}{y}$
- , and solve for
- y
- if possible.

Example 9.5. Solve $t^3 + y^3 - ty^2y' = 0$ twice, using each of the substitutions $y = ut$ and $t = vy$. Assume that $t > 0$ and $y > 0$.

Example 9.6. Solve $ty - y^2 + t(t - 3y)y'$.

Technique 9.7 (Summary of Techniques for Solving First-Order ODEs).

- (1) **Separable Equations** - Of the form $g(y)y' = f(t)$. To solve: **Separate and integrate.**
- (2) **Linear Equations** - Of the form $y + p(t)y' = q(t)$. To solve: **Multiply by an integrating factor.**
- (3) **Exact Equations** - Of the form $M(t, y) + N(t, y)y' = 0$ where M, N are the partial derivatives of some potential function. To solve: **Find a potential function.**
- (4) **Bernoulli Equations** - Of the form $y' + p(t)y = q(t)y^n$. To solve: **Substitute and reduce to the linear case.**
- (5) **Homogeneous Equations** - Of the form $M(t, y) + N(t, y)y' = 0$ where M, N are homogeneous. To solve: **Substitute and reduce to the separable case.**

10 Applications: Population Growth

Definition 10.1. The **Malthus model** for population growth is the IVP discussed in Example 5.2, i.e.

$$y' = ky, y(0) = y_0,$$

where y is population at time t , and y_0 is some initial population. Recall that the solution to the above IVP is

$$y = y_0 e^{kt}.$$

Despite the simplicity and effectiveness of the Malthus model, it has drawbacks. Primarily, the Malthus model assumes that population growth is always directly proportional to population size, i.e. $y' = ky$ for constant k , and thus all population growth must be exponential. This is a situation that is usually not encountered in the long term in reality, as typically the growth of a population depends on the natural resources available in a given system. One way to try to reflect this reality is by changing our fundamental assumption: we still assume that the growth y' is proportional to the total population y , but now as the population y grows larger, the population growth will be inhibited due to scarcity of resources. In other words, instead of assuming that y' is a constant k times population size at any time, we assume that y' is a function $(k - ay)$ times population size, where k and a are constant; as y increases the proportion of y' to y decreases. This new assumption leads to consideration of the following famous differential equation.

Definition 10.2. The **logistic equation** is the ODE

$$y' = (k - ay)y = ky - ay^2,$$

where k and a are constant.

Example 10.3. Solve the IVP given by the logistic equation and the initial condition $y(0) = y_0$.

Solution. The logistic equation is autonomous, hence separable. The usual methods yield the following general solution:

$$y = \frac{ky_0}{ay_0 + (k - ay_0)e^{-kt}}.$$

□

Example 10.4. Show that if y is a solution to the logistic equation, then y is increasing but $\lim_{t \rightarrow \infty} y$ is finite.

Definition 10.5. Let y be a solution to the logistic equation with given constants k and a . Then the number $\frac{k}{a} = \lim_{t \rightarrow \infty} y$ is the **maximum sustainable population** of the model.

11 Applications: Newton's Law of Cooling

Theorem 11.1 (Newton's Law of Cooling). *The rate at which the temperature T changes with respect to time t in a cooling body is proportional to the difference $T - T_s$ between the temperature of the body T and the constant temperature T_s of the surrounding medium. In other words,*

$$T' = k(T - T_s) \text{ for some constant } k < 0.$$

Example 11.2. (1) Find a solution to the IVP $T' = k(T - T_s)$, $T(0) = T_0$.
(2) Compute $\lim_{t \rightarrow \infty} T$.

Example 11.3. A pie is removed from a 350° Fahrenheit oven and placed to cool in a room with temperature 75° Fahrenheit. In 15 minutes, the pie has a temperature of 150° Fahrenheit. Determine the time required to cool the pie to a temperature of 80° .

12 Applications: Mixing Problems

Example 12.1. Suppose a tank holds V_0 gallons of a brine solution. Another brine solution, with concentration S_1 pounds of salt per gallon, is allowed to flow into the tank at a rate R_1 gallons per minute. A drain at the bottom of the tank allows the mixture in the tank to flow out at a rate of R_2 gallons per minute. We assume the brine flowing out is well-mixed. If y represents the number of pounds of salt in the tank after t minutes, then y is modeled by the ODE:

$$y' = (\text{Rate salt enters the tank}) - (\text{Rate salt leaves the tank}) = S_1 R_1 - \frac{y(t)}{V(t)} \cdot R_2,$$

where $V(t)$ is the volume of liquid in the tank at time t . This function V is modelled by the ODE

$$V' = R_1 - R_2.$$

Example 12.2. Suppose $S_1 = 1$, $R_1 = 4$, and $R_2 = 3$. In addition, suppose the tank initially holds 500 gallons of liquid and 250 pounds of salt.

- (1) Find the amount of salt contained in the tank at any time.
- (2) Determine how much time is required for 1000 pounds of salt to accumulate.
- (3) If the tank holds a maximum of 800 gallons, can 1000 pounds of salt be present in the tank at any time before the tank reaches its maximum capacity?

13 Complex Numbers and Euler's Formula

Definition 13.1. A **complex number** is a number of the form $z = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. The **real part** of z is the real number $\operatorname{Re} z = a$, and the **imaginary part** of z is the real number $\operatorname{Im} z = b$. We denote the set of all complex numbers by \mathbb{C} .

We visualize the complex numbers as a set by identifying each complex number $a + bi$ with the point (a, b) in \mathbb{R}^2 ; this gives a one-to-one correspondence between \mathbb{C} and \mathbb{R}^2 and leads us to commonly refer to \mathbb{C} as the **complex plane**.

Utilizing this visual representation, the **modulus** of a complex number $z = a + bi$, denoted $|z|$, is the Cartesian distance from (a, b) to the origin, i.e.

$$|z| = \sqrt{a^2 + b^2}.$$

In addition, by identifying \mathbb{C} with \mathbb{R}^2 , we can formally talk about limits, continuity, etc. in \mathbb{C} by declaring that a limit exists in \mathbb{C} if and only if the limit of the associated points exists in \mathbb{R}^2 (limits in \mathbb{R}^2 were studied in the student's multivariable calculus course).

Fact 13.2. Given any non-zero complex number $z = a + bi$, there exists a unique angle θ , with $0 \leq \theta < 2\pi$, called the **argument** of z , for which

$$z = |z|(\cos \theta + i \sin \theta).$$

Definition 13.3. Given two complex numbers $z = a + bi$ and $w = c + di$, one can define their sum $z + w$ and their product zw by extending the addition and multiplication of the reals in the natural way. The sum $z + w$ is defined to be:

$$z + w = (a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i.$$

By associating $a + bi$ and $c + di$ in \mathbb{C} with the points (a, b) and (c, d) in \mathbb{R}^2 in the usual way, it is easy to see that complex addition corresponds exactly to vector addition in \mathbb{R}^2 . However, complex multiplication (defined below) clearly distinguishes \mathbb{C} from \mathbb{R}^2 :

$$zw = (a + bi)(c + di) = ac + adi + cbi + bdi^2 = (ac - bd) + (ad + bc)i.$$

It is straightforward to verify that the usual commutative properties of addition and multiplication, associative properties of addition and multiplication, and distributive property of multiplication are true in \mathbb{C} just like they are in \mathbb{R} .

Example 13.4. Let $z = 1 + i$ and $w = 2 + 3i$. Compute $z + w$ and zw .

Definition 13.5. The **complex conjugate** of a complex number $z = a + bi$, denoted \bar{z} , is the number $\bar{z} = a - bi$.

Example 13.6. Show that for any complex number z , we have $z\bar{z} = |z|^2$.

Example 13.7. Show that for any complex number $z = a + bi$ and any complex number $w = c + di \neq 0$, the quotient $\frac{z}{w}$ exists and is a unique complex number. Therefore division makes sense in \mathbb{C} .

Initially the operations of multiplication and division in \mathbb{C} may seem fairly mysterious. We will understand them better after we observe Euler's formula below, but in order to get Euler's formula we need to recall the following facts, developed in a previous calculus course.

Fact 13.8 (Maclaurin Series for e^x , $\sin x$, and $\cos x$). *The functions e^x , $\sin x$, and $\cos x$ have the following Maclaurin series representations:*

$$\begin{aligned} (1) \quad e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ (2) \quad \sin x &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ (3) \quad \cos x &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

Moreover each of the series above have interval of convergence $(-\infty, \infty) = \mathbb{R}$. In other words, the equalities above hold for all real numbers x .

Until now we have always regarded the functions e^x , $\sin x$, and $\cos x$ as real-valued functions of a real variable. The following fact says that there is a meaningful way to extend the functions e^x , $\sin x$, and $\cos x$ to be complex-valued functions of a complex variable, that is, to have domain \mathbb{C} and codomain \mathbb{C} . This fact is incredibly useful and non-obvious, and the proof is not very difficult to understand, but is a little beyond the scope of our course and so we omit it. (One typically sees this proof in an undergraduate complex analysis course.)

Fact 13.9. *Each of the Maclaurin series in Fact 13.8 above converges not just for all real numbers x in \mathbb{R} , but in fact for all complex numbers x in \mathbb{C} . Of course if x is real then e^x , $\sin x$, and $\cos x$ are real, but if x is complex non-real then e^x , $\sin x$, and $\cos x$ are complex non-real.*

Moreover, the function e^x , now regarded as a complex function of a complex variable, still obeys the exponentiation rule: for any two complex numbers z and w , $e^{z+w} = e^z e^w$.

Theorem 13.10 (Euler's Formula). *For all real numbers x ,*

$$e^{ix} = \cos x + i \sin x.$$

Proof. The proof is by computation using the previous two facts:

$$\begin{aligned} e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots \\ &= \left(1 + \frac{i^2 x^2}{2!} + \frac{i^4 x^4}{4!} + \frac{i^6 x^6}{6!} + \dots \right) + i \left(x + \frac{i^2 x^3}{3!} + \frac{i^4 x^5}{5!} + \frac{i^6 x^7}{7!} + \dots \right) \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\ &= \cos x + i \sin x, \end{aligned}$$

□

Now it is time to harvest the corollaries. The first is a cute fact relating some of our favorite mathematical constants.

Corollary 13.11. $e^{\pi i} = -1$.

Proof. Take $x = \pi$ in Euler's formula. □

The next two shed a great deal of light on the connection between the planar representation of \mathbb{C} and the multiplication in \mathbb{C} .

Corollary 13.12. Suppose z is a complex number with argument θ . Then $z = |z|e^{i\theta}$.

Corollary 13.13. Let z be a complex number with argument θ_1 , so $z = |z|e^{i\theta_1}$, and let w be a complex number with argument θ_2 , so $w = |w|e^{i\theta_2}$. Then the product zw is uniquely determined by the following facts:

(1) zw has modulus $|zw| = |z||w|$; and

(2) zw has argument $\theta_1 + \theta_2$.

Proof. Compute: $zw = |z|e^{i\theta_1}|w|e^{i\theta_2} = |z||w|e^{i\theta_1+i\theta_2} = |z||w|e^{i(\theta_1+\theta_2)}$. □

14 Second-Order Linear Homogeneous Equations with Constant Coefficients

Definition 14.1. Recall that a **second-order linear ODE** is of the form

$$y'' + p(t)y' + q(t)y = f(t).$$

Recall also that a second-order linear ODE in the above form is **homogeneous** if $f(t) = 0$. A second-order linear homogeneous equation **has constant coefficients** if $p(t) = p$ and $q(t) = q$ are just constants.

Example 14.2. Consider the ODE $y'' - 2y' - 15y = 0$.

- (1) Show that the equation has at least two solutions of the form $y = e^{\lambda_1 t}$, $y = e^{\lambda_2 t}$, where λ_1 and λ_2 are real numbers.
- (2) Show that any function of the form $y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, where C_1 and C_2 are arbitrary constants, are also solutions to the ODE.

Solution - Characteristic Polynomial with Distinct Real Roots. (a) Let us drop subscripts for a moment and consider functions of the form $y = e^{\lambda t}$. Differentiating once and twice, we get

$$\begin{aligned} y' &= \lambda e^{\lambda t}; \\ y'' &= \lambda^2 e^{\lambda t}. \end{aligned}$$

Plugging these into the ODE, we get

$$\lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} - 15e^{\lambda t} = (\lambda^2 - 2\lambda - 15)e^{\lambda t} = 0.$$

Judging from the above, $y = e^{\lambda t}$ will be a solution if either $\lambda^2 - 2\lambda - 15 = 0$ or $e^{\lambda t} = 0$. Since the latter equality never happens ($e^{\lambda t}$ is always positive), we need λ to be a root of the quadratic polynomial $\lambda^2 - 2\lambda - 15 = (\lambda + 3)(\lambda - 5)$. So it suffices to take $\lambda_1 = -3$ and $\lambda_2 = 5$, i.e. $y = e^{-3t}$ and $y = e^{5t}$ are solutions to the ODE (this is easy to verify directly).

(b) Suppose $y_1 = e^{-3t}$ and $y_2 = e^{5t}$, so y_1 and y_2 are solutions to the ODE, and C_1 and C_2 are arbitrary constants. Then taking $y = C_1 y_1 + C_2 y_2$, we get

$$\begin{aligned}
y'' - 2y' - 15y = 0 &= (C_1y_1 + C_2y_2)'' - 2(C_1y_1 + C_2y_2)' - 15(C_1y_1 + C_2y_2) \\
&= C_1(y_1'' - 2y_1' - 15y_1) + C_2(y_2'' - 2y_2' - 15y_2) \\
&= C_1 \cdot 0 + C_2 \cdot 0 = 0.
\end{aligned}$$

So $y = C_1y_1 + C_2y_2$ is a solution as well. \square

In the above example, given a second-order linear homogeneous ODE with constant coefficients, we found infinitely many solutions of the form $y = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$. We will see in the next section that in fact all of its solutions have this form.

The example and remarks above suggest that given a second-order linear homogeneous ODE with constant coefficients $ay'' + by' + cy = 0$, the task of finding solutions $y = e^{\lambda t}$ may reduce in general to the task of finding solutions to the associated polynomial equation $a\lambda^2 + b\lambda + c = 0$.

Definition 14.3. Given a second-order linear homogeneous ODE with constant coefficients $ay'' + by' + cy = 0$, the associated **characteristic polynomial** is

$$a\lambda^2 + b\lambda + c.$$

The **characteristic equation** is

$$a\lambda^2 + b\lambda + c = 0.$$

Theorem 14.4. Let $ay'' + by' + cy = 0$ be a second-order linear homogeneous ODE with constant coefficients. If r is a solution to the characteristic equation, then $y = e^{rt}$ is a solution to the ODE.

In our previous example, the characteristic polynomial happened to have two distinct real roots; what happens if the polynomial has complex roots, or a single repeated real root? The next two examples (and their preceding observations) address these situations.

Theorem 14.5. Let $ay'' + by' + cy = 0$ be a second-order linear homogeneous ODE with constant coefficients. A complex-valued function y is a solution to the ODE if and only if both $\operatorname{Re} y$ and $\operatorname{Im} y$ are solutions.

Proof. Write $f = \operatorname{Re} y$ and $g = \operatorname{Im} y$, so $y = f + gi$. Plugging in and using the linearity of the derivative, we get

$$ay'' + by' + cy = a(f'' + g''i) + b(f' + g'i) + c(f + gi) = (af'' + bf' + cf) + (ag'' + bg' + cg)i = 0.$$

The left-hand side will be 0 if and only if both the real part and the complex part of the right-hand side are 0. So y is a solution if and only if both f and g are. \square

Example 14.6. Solve $y'' + 4y' + 20y = 0$.

Solution - Characteristic Polynomial with Complex Conjugate Roots. We proceed as in the previous example by assuming that y has solutions of the form $y = e^{\lambda t}$. Differentiating to find $y' = \lambda e^{\lambda t}$ and $y'' = \lambda^2 e^{\lambda t}$, we can plug into the differential equation and solve to obtain (as in the previous example):

$$(\lambda^2 + 4\lambda + 20)e^{\lambda t} = 0.$$

Since $e^{\lambda t}$ above is never zero, we have $\lambda^2 + 4\lambda + 20 = 0$ and we wish to solve for λ . The quadratic formula gives us:

$$\lambda = -2 + 4i \text{ or } \lambda = -2 - 4i.$$

We may safely conclude that the *complex-valued functions of a complex variable* $y = e^{(-2+4i)t}$ and $y = e^{(-2-4i)t}$ are solutions to the given differential equation. But we are interested in finding only *real-valued functions of a real variable* which satisfy the ODE. Now given a complex function y , it is easy to check that y satisfies a given differential equation if and only if both its real part $\operatorname{Re} y$ and its imaginary part $\operatorname{Im} y$ satisfy the same equation. Thus to find some real-valued solutions, it suffices to take the real and complex parts of our complex solutions using Euler's formula. Recall that

$$e^{(-2+4i)t} = e^{-2t} e^{(4t)i} = e^{-2t}(\cos(4t) + i \sin(4t))$$

and likewise

$$e^{(-2-4i)t} = e^{-2t} e^{(-4t)i} = e^{-2t}(\cos(-4t) + i \sin(-4t)) = e^{-2t}(\cos(4t) - i \sin(4t))$$

Thus both of the functions $y = e^{-2t} \cos(4t)$ and $y = e^{-2t} \sin(4t)$ are solutions of the equation. (The student should verify this manually.) And as in the previous example, it will turn out that all solutions to the ODE are of the form

$$y = C_1 e^{-2t} \cos(4t) + C_2 e^{-2t} \sin(4t)$$

where C_1 and C_2 are arbitrary constants. (We will observe this rigorously in the next section.) \square

Theorem 14.7. *Let $ay'' + by' + cy = 0$ be a second-order linear homogeneous ODE with constant coefficients. Suppose the characteristic polynomial has a single repeated real root, i.e. $a\lambda^2 + b\lambda + c = a(\lambda - r)^2$ for some real number r . Then $y = e^{rt}$ is a solution, and $y = te^{rt}$ is also a solution.*

Proof. The fact that $y = e^{rt}$ is a solution follows from what we've already done. The fact that $y = te^{rt}$ is a clever non-obvious observation. Let us see that it's true. Compute by the product rule:

$$y' = rte^{rt} + e^{rt} = (rt + 1)e^{rt}$$

and

$$y'' = r(rt + 1)e^{rt} + re^{rt} = r(rt + 1 + 1)e^{rt} = (r^2t + 2r)e^{rt}.$$

Now plugging into the ODE we get

$$\begin{aligned} ay'' + by' + cy &= [a(r^2t + 2r) + b(rt + 1) + ct]e^{rt} \\ &= [(ar^2 + br + c)t + (2ra + b)]e^{rt} \\ &= [(ar^2 + br + c)t + a(2r + \frac{b}{a})]e^{rt}. \end{aligned}$$

Now $ar^2 + br + c = 0$ since r is a root of the characteristic polynomial. Even better, $-2r = \frac{b}{a}$ since r is a repeated root (to see this, foil out the right-hand side of the equality $\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = (\lambda - r)^2$), so $a(2r + \frac{b}{a}) = 0$. Thus the right-hand side above equals 0, showing that $y = te^{rt}$ is a solution. \square

Example 14.8. Solve $y'' + 2y' + 1 = 0$.

Solution - Characteristic Polynomial with Real Repeated Root. Once again assume there are solutions of the form $y = e^{\lambda t}$. Differentiate, plug in and solve (see previous examples) to reduce to the following equation:

$$\lambda^2 + 2\lambda + 1 = 0.$$

The above factors into $(\lambda + 1)^2 = 0$, and thus has only one real solution $\lambda = -1$. So $y = e^{-t}$ and $y = te^{-t}$ are solutions. We conclude that $y = C_1e^{-t} + C_2te^{-t}$ is a general solution to the ODE. \square

Technique 14.9 (General Method for Solving Second-Order Linear Homogeneous ODE's with Constant Coefficients).

Given an ODE of the form $ay'' + by' + c = 0$, solve the associated characteristic equation $a\lambda^2 + b\lambda + c = 0$.

- (2) If the characteristic equation has two distinct real solutions λ_1 and λ_2 , then a general solution to the ODE is given by $y = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$.
- (3) If the characteristic equation has one repeated real solution λ , then a general solution to the ODE is given by $y = C_1e^{\lambda t} + C_2te^{\lambda t}$.
- (4) If the characteristic equation has two complex conjugate solutions $z = a + bi$ and $\bar{z} = a - bi$, then a general solution to the ODE is given by $y = C_1e^{at} \cos(bt) + C_2e^{at} \sin(bt)$.

15 The Wronskian and the Structure of Solutions to Second-Order Linear Homogeneous ODEs

The main purpose of this section will be to justify the fact that the solution sets given in Technique 14.9 in fact encompass all possible solutions to a given second-order linear homogeneous ODE. First we will state without proof an analogue of the Picard-Lindelöf theorem for second-order differential equations.

Theorem 15.1 (Existence and Uniqueness of Solutions for Second-Order Linear Equations). *Suppose that $p(t)$, $q(t)$, and $f(t)$ are continuous functions on an open interval I containing t_0 , and consider the IVP*

$$y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

Then there exists one and only one solution to the IVP on the interval I .

As we proceed, we will need to recall some terminology from linear algebra.

Definition 15.2. Let f_1, f_2, \dots, f_n be a finite collection of functions of an independent variable t . If c_1, c_2, \dots, c_n are any constants, then the function

$$c_1f_1 + c_2f_2 + \dots + c_nf_n$$

is called a **linear combination** of the functions f_1, f_2, \dots, f_n .

The functions f_1, \dots, f_n are called **linearly dependent** if there exist constants c_1, c_2, \dots, c_n , at least one of which is non-zero, for which

$$c_1f_1 + c_2f_2 + \dots + c_nf_n = 0.$$

If f_1, \dots, f_n are not linearly dependent, they are called **linearly independent**.

Example 15.3. Determine if the following pairs of functions are linearly dependent or linearly independent.

- (1) $f_1(t) = 2t; f_2(t) = 4t$

- (2) $f_1(t) = e^t; f_2(t) = e^{-2t}$

$$(3) f_1(t) = \sin 2t, f_2(t) = 5 \sin t \cos t$$

$$(4) f_1(t) = t, f_2(t) = 0$$

Theorem 15.4. Two functions f_1 and f_2 are linearly dependent if and only if one is a constant multiple of the other. □

Theorem 15.5 (Principle of Superposition). Suppose f_1 and f_2 are two solutions to the linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ on the interval I , and let c_1 and c_2 be any constants. Then the linear combination $c_1f_1(t) + c_2f_2(t)$ is also a solution on the interval I .

Proof. Plug it in and see that it works! □

Definition 15.6. Let $f_1(t)$ and $f_2(t)$ be two differentiable functions of an independent variable t . The **Wronskian** of f_1 and f_2 is the function

$$W(f_1, f_2)(t) = \det \begin{bmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{bmatrix}.$$

Example 15.7. Compute the Wronskian of the following functions.

- (1) $\cos t$ and $\sin t$.
- (2) e^t and te^t .
- (3) $2t$ and $4t$.
- (4) $\sin(2t)$ and $5 \sin t \cos t$.

Theorem 15.8. Let y_1 and y_2 be two solutions to the ODE $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous. Then y_1 and y_2 are linearly dependent if and only if $W(y_1, y_2) = 0$, i.e. $W(y_1, y_2)$ is identically the zero function.

Proof. If y_1 and y_2 are linearly dependent, then there is a constant C for which $y_2 = Cy_1$. In this case $W(y_1, y_2) = y_1y_2' - y_1'y_2 = y_1(Cy_1') - y_1'(Cy_1) = 0$.

Conversely, suppose $W(y_1, y_2) = 0$. Choose any point t_0 in the domain of y_1 and y_2 . Since the Wronskian is 0, we have $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) = 0$ and hence

$$\frac{y_1(t_0)}{y_2(t_0)} = \frac{y_1'(t_0)}{y_2'(t_0)}.$$

Let $C = \frac{y_1(t_0)}{y_2(t_0)} = \frac{y_1'(t_0)}{y_2'(t_0)}$. Now consider the function $y = y_1 - Cy_2$. By the Principle of Superposition Theorem 15.5, y is a solution to the given ODE. In addition, notice that

$$y(t_0) = y_1(t_0) - Cy_2(t_0) = 0 \text{ and } y'(t_0) = y_1'(t_0) - Cy_2'(t_0) = 0.$$

In other words, $y = y_1 - Cy_2$ is a solution to the IVP $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = 0$, $y'(t_0) = 0$. But the constant zero function 0 is also a solution to this IVP! Therefore by the Existence and Uniqueness Theorem 15.1, we must have $y_1 - Cy_2 = 0$. It follows that $y_1 = Cy_2$, whence y_1 and y_2 are a linearly dependent pair, as claimed. □

Lemma 15.9. Let y_1 and y_2 be functions. Then $\frac{d}{dt}W(y_1, y_2) = y_1y_2'' - y_1''y_2$.

Proof.

$$\begin{aligned} \frac{d}{dt}W(y_1, y_2) &= \frac{d}{dt}(y_1y_2' - y_1'y_2) \\ &= (y_1y_2'' + y_1'y_2') - (y_1'y_2' + y_1''y_2) \\ &= y_1y_2'' - y_1''y_2. \end{aligned}$$

□

Lemma 15.10 (Abel's Theorem). *Let y_1 and y_2 be two solutions of the second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$, where p and q are continuous functions of t . Let $P(t)$ be an antiderivative of $p(t)$. If y_1 and y_2 are linearly independent, then*

$$W(y_1, y_2)(t) = Ce^{-P(t)},$$

where C is a non-zero constant.

Proof. Suppose y_1 and y_2 are two independent solutions to $y'' + p(t)y' + q(t)y = 0$. Since y_1 and y_2 are solutions to the ODE, we have (omitting arguments):

$$qy_1 = -y_1'' - py_1' \quad \text{and} \quad qy_2 = -y_2'' - py_2'.$$

Therefore,

$$\begin{aligned} 0 &= y_2 \cdot qy_1 - y_1 \cdot qy_2 \\ &= y_2(-y_1'' - py_1') - y_1(-y_2'' - py_2') \\ &= (y_1y_2'' - y_1''y_2) + p(y_1y_2' - y_1'y_2) \\ &= \frac{d}{dt}W(y_1, y_2) + pW(y_1, y_2). \end{aligned}$$

The last line above follows from the definition of the Wronskian plus Lemma 15.9. So the above computation yields a separable ODE in t and $W(t) = W(y_1, y_2)(t)$:

$$W' + p(t)W = 0.$$

Let $P(t)$ be some antiderivative of $p(t)$, so $P' = p$. By Theorem 6.6, all solutions to the ODE above have the form $W(t) = Ce^{-P(t)}$, where C is a constant. Since y_1 and y_2 are independent, by Theorem 15.8, we must have $C \neq 0$. \square

Corollary 15.11. *Let y_1 and y_2 be two solutions of the second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$, where p and q are continuous functions of t . If y_1 and y_2 are linearly independent, then $W(y_1, y_2)(t) \neq 0$ for all t , i.e. $W(y_1, y_2)$ is nowhere zero.*

Theorem 15.12 (General Solutions to Second-Order Linear Homogeneous ODE's). *Let y_1 and y_2 be two linearly independent solutions to the second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous functions of t . Then all solutions to the ODE are of the form $y = C_1y_1 + C_2y_2$ for some constants C_1 and C_2 .*

Proof. Let Y be an arbitrary solution to the ODE; we will in fact compute the constants C_1 and C_2 for which $Y = C_1y_1 + C_2y_2$! Fix any real number t_0 in the domain of Y . By Corollary 15.11, we have $W(y_1, y_2)(t_0) \neq 0$. Therefore it makes sense to set:

$$C_1 = \frac{W(Y, y_2)(t_0)}{W(y_1, y_2)(t_0)}$$

and

$$C_2 = \frac{W(y_1, Y)(t_0)}{W(y_1, y_2)(t_0)}.$$

Set $Z = C_1y_1 + C_2y_2$. Note that Z is a solution to the ODE by the Principle of Superposition Theorem 15.5. If we can show $Y = Z$, then we will have shown that Y is a linear combination of y_1 and y_2 , and hence the proof will be complete. Let us compute $Z(t_0)$ and $Z'(t_0)$:

$$\begin{aligned} Z(t_0) &= C_1y_1(t_0) + C_2y_2(t_0) \\ &= \frac{W(Y, y_2)(t_0)}{W(y_1, y_2)(t_0)} \cdot y_1(t_0) + \frac{W(y_1, Y)(t_0)}{W(y_1, y_2)(t_0)} y_2(t_0) \\ &= \frac{(Y(t_0)y_2'(t_0) - Y'(t_0)y_2(t_0))y_1(t_0) - (y_1(t_0)Y'(t_0) - y_1'(t_0)Y(t_0))y_2(t_0)}{W(y_1, y_2)(t_0)} \\ &= \frac{Y(t_0)(y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0))}{W(y_1, y_2)(t_0)} \\ &= Y(t_0). \end{aligned}$$

$$\begin{aligned} Z'(t_0) &= C_1y_1'(t_0) + C_2y_2'(t_0) \\ &= \frac{W(Y, y_2)(t_0)}{W(y_1, y_2)(t_0)} \cdot y_1'(t_0) + \frac{W(y_1, Y)(t_0)}{W(y_1, y_2)(t_0)} y_2'(t_0) \\ &= \frac{(Y(t_0)y_2'(t_0) - Y'(t_0)y_2(t_0))y_1'(t_0) - (y_1(t_0)Y'(t_0) - y_1'(t_0)Y(t_0))y_2'(t_0)}{W(y_1, y_2)(t_0)} \\ &= \frac{Y'(t_0)(y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0))}{W(y_1, y_2)(t_0)} \\ &= Y'(t_0). \end{aligned}$$

Thus we have shown that Y and Z are two solutions to the ODE which agree on the initial conditions $y(t_0) = Y(t_0) = Z(t_0)$ and $y'(t_0) = Y'(t_0) = Z'(t_0)$. But by the Existence and Uniqueness Theorem 15.1, there is only one such function! Therefore $Y = Z$, and this completes the proof. \square

The theorem above justifies the general solutions we gave in Technique 14.9, and motivates the following definition.

Definition 15.13. A pair of functions $\{y_1, y_2\}$ is called a **fundamental set of solutions** to a linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ if y_1 and y_2 are both solutions and if they are linearly independent. If $\{y_1, y_2\}$ is a fundamental set of solutions, then all solutions to the ODE are linear combinations of y_1 and y_2 .

16 Second Order Linear Non-Homogeneous ODE's: The Method of Undetermined Coefficients

Definition 16.1. Given a second-order linear ODE $y'' + p(t)y' + q(t)y = f(t)$, the **corresponding homogeneous equation** is $y'' + p(t)y' + q(t)y = 0$.

The next theorem reduces the task of finding a general solution to a second-order linear ODE to the task of finding a single solution, together with solving the corresponding homogeneous equation.

Theorem 16.2. If y_1 and y_2 are solutions to the second-order linear ODE $y'' + p(t)y' + q(t)y = f(t)$, then

$$y_2 = y_1 + y_h,$$

where y_h is some solution to the corresponding homogeneous equation. In other words, two solutions differ only by a solution to the corresponding homogeneous equation.

Proof. This proof mimics exactly the proof we gave long ago in the first-order case. With y_1 and y_2 given, set $y_h = y_2 - y_1$. Then plug y_h into the corresponding homogeneous equation and see that it is a solution. \square

Example 16.3. Consider the ODE $y'' - 4y = -3 \sin t$.

- (1) Verify that $y_p(t) = \frac{3}{5} \sin t$ is a solution.
- (2) Find all solutions.

Solution. Checking (a) is easy, so we proceed with (b). Since we know a particular solution (from part (a)), by the previous theorem we can find all possible solutions simply by finding all solutions to the corresponding homogeneous equation:

$$y'' - 4y = 0.$$

For this we consider the characteristic equation $\lambda^2 - 4 = 0$. This has two distinct roots 2 and -2 , so a general solution is given by

$$y = C_1 e^{2t} + C_2 e^{-2t}.$$

Therefore all possible solutions to the original ODE have the form

$$y = \frac{3}{5} \sin t + C_1 e^{2t} + C_2 e^{-2t}.$$

\square

The next technique is much more easily understood via practice than by reading the description, but we will attempt to describe the process in words anyway. See the examples following the technique.

Technique 16.4 (Guessing a Particular Solution to a Second-Order Linear Non-Homogeneous ODE with Constant Coefficients). Given an ODE of the form $ay'' + by' + cy = f(t)$, we look at the function $f(t)$ on the right-hand side. This method is only effective if $f(t)$ is a linear combination of functions of the forms t^m , $t^m e^{\alpha t}$, $t^m e^{\alpha t} \cos(\beta t)$, and $t^m e^{\alpha t} \sin(\beta t)$, where m is a positive integer and α, β are real numbers.

- (1) If f is indeed a linear combination of functions of this form, then we consider the following sets of functions:
 - (a) $F = \{1, t, t^2, \dots, t^m\}$ if t^m appears in f ;
 - (b) $F = \{e^{\alpha t}, t e^{\alpha t}, t^2 e^{\alpha t}, \dots, t^m e^{\alpha t}\}$ if $t^m e^{\alpha t}$ appears in f ; and
 - (c) $F = \{e^{\alpha t} \cos(\beta t), t e^{\alpha t} \cos(\beta t), \dots, t^m e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t), t e^{\alpha t} \sin(\beta t), \dots, t^m e^{\alpha t} \sin(\beta t)\}$ if $t^m e^{\alpha t} \cos(\beta t)$ or $t^m e^{\alpha t} \sin(\beta t)$ appears in f .
- (2) If none of the functions listed above appear as solutions to the corresponding homogeneous equation, then we guess that a particular solution y_p to the equation will be a linear combination of the functions as above.
- (3) On the other hand, if some functions in F appear as solutions to the corresponding homogeneous equation, then we multiply F by t^r to obtain a new set $t^r F$, where r is the smallest integer for which no function in $t^r F$ is a solution to the corresponding homogeneous equation. In this case we guess that y_p is a linear combination of the functions above, but with F replaced by $t^r F$.

Technique 16.5 (Method of Undetermined Coefficients). Given an ODE of the form $ay'' + by' + cy = f(t)$:

- (1) Solve the corresponding homogeneous equation, to obtain a general solution y_h .

- (2) Guess the form of a particular solution y_p to the ODE, as in the previous technique.
- (3) With y_p written as a general linear combination (with undetermined coefficients) of functions, compute y_p' and y_p'' and plug into the ODE. Equate the coefficients of like terms, and solve for the coefficients. This gives a particular solution y_p .
- (4) A general solution to the original ODE is $y_p + y_h$ (where y_p is a particular function, and y_h denotes infinitely many functions).

Example 16.6. Solve $y'' + 5y' + 6y = 2e^t + 4 \cos t$.

Solution. First we go ahead and solve the corresponding homogeneous equation:

$$y'' + 5y' + 6y = 0.$$

By the usual method of considering the characteristic equation, it is easy to see that a fundamental set of solutions to the corresponding homogeneous equation is $\{e^{2t}, e^{3t}\}$. So a general solution is of the form $y_h = C_1 e^{2t} + C_2 e^{3t}$.

We note the $2e^t$ and the $4 \cos t$ on the right-hand side, and we guess that a particular solution y_p will be a linear combination of the functions e^t , $\cos t$, and $\sin t$. (Because none of these functions is a solution to the corresponding homogeneous equation, there are no wrinkles here- for remarks on what to do if one of our guesses appears as a solution to the corresponding homogeneous equation, see the next example). In other words, we guess that

$$y_p = Ae^t + B \cos t + C \sin t,$$

where A , B , and C are constants to be determined. We compute derivatives:

$$\begin{aligned} y_p' &= Ae^t - B \sin t + C \cos t \\ y_p'' &= Ae^t - B \cos t - C \sin t \end{aligned}$$

and plug into the ODE:

$$Ae^t - B \cos t - C \sin t + 5(Ae^t - B \sin t + C \cos t) + 6(Ae^t + B \cos t + C \sin t) = 2e^t + 4 \cos t.$$

Regrouping terms, we get:

$$3Ae^t + C \cos t - B \sin t = 2e^t + 4 \cos t.$$

We conclude that $A = \frac{2}{3}$, $C = 4$, and $B = 0$. So $y_p(t) = \frac{2}{3}e^t + 4 \sin t$ is a particular solution. We conclude that a general solution to our given ODE is any function of the form

$$y = \frac{2}{3}e^t + 4 \sin t + C_1 e^{2t} + C_2 e^{3t}.$$

□

Example 16.7. Solve the IVP $y'' - 3y' = -e^{3t} - 2t$, $y(0) = 0$, $y'(0) = \frac{8}{9}$.

Solution. First we solve the corresponding homogeneous equation $y'' - 3y' = 0$ by considering the characteristic equation $\lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$. This has roots 0 and 3, so a fundamental set of solutions is $\{e^{0t}, e^{3t}\} = \{1, e^{3t}\}$, and a general solution has the form

$$y_h = C_1 + C_2 e^{3t}.$$

Now looking at the right-hand side of our original ODE, we are inclined to guess that a particular solution y_p will look like some linear combination of the functions e^{3t} , 1, and t . However, e^{3t} and 1 already appear as solutions to the corresponding homogeneous equation; therefore, we multiply everything by t (in other words by t^1 , where 1 is the least power which will prevent all functions from being solutions to the corresponding homogeneous equation). So we guess that y_p is a linear combination of the functions te^{3t} , t , and t^2 :

$$y_p(t) = Ate^{3t} + Bt + Ct^2$$

for some undetermined coefficients A, B, C . Now compute derivatives:

$$y_p'(t) = Ae^{3t} + 3Ate^{3t} + B + 2Ct$$

$$\begin{aligned} y_p''(t) &= 3Ae^{3t} + 3Ae^{3t} + 9Ate^{3t} + 2C \\ &= 6Ae^{3t} + 9Ate^{3t} + 2C. \end{aligned}$$

Plugging these into the ODE we get

$$6Ae^{3t} + 9Ate^{3t} + 2C - 3(Ae^{3t} + 3Ate^{3t} + B + 2Ct) = -e^{3t} - 2t.$$

Regrouping, we have

$$3Ae^{3t} + (2C - 3B) - 6Ct = -e^{3t} - 2t.$$

It follows from the above that $3A = -1$ and $-6C = -2$, whence $A = -\frac{1}{3}$ and $C = \frac{1}{3}$. Since $2C - 3B = 0$, we can also deduce that $B = \frac{2}{9}$. So a particular solution to the ODE is given by

$$y_p = -\frac{1}{3}te^{3t} + \frac{2}{9}t + \frac{1}{3}t^2.$$

We conclude that any solution to the ODE must have the form

$$y = -\frac{1}{3}te^{3t} + \frac{2}{9}t + \frac{1}{3}t^2 + C_1 + C_2e^{3t}.$$

Now it remains only to solve for the initial conditions. We first pin down C_2 by considering y' :

$$y'(t) = -te^{3t} + (3C_2 - \frac{1}{3})e^{3t} + \frac{2}{3}t + \frac{2}{9}.$$

Our initial condition gives us

$$\frac{8}{9} = y'(0) = 3C_2 - \frac{1}{3} + \frac{2}{9}$$

whence $C_2 = \frac{1}{3}$. Now we can use our other initial condition $y(0) = 0$ to find C_1 :

$$0 = y(0) = C_1 + C_2 = C_1 + \frac{1}{3}$$

The above gives $C_1 = -\frac{1}{3}$. So the unique solution to our IVP is

$$y(t) = -\frac{1}{3}te^{3t} + \frac{2}{9}t + \frac{1}{3}t^2 - \frac{1}{3} + \frac{1}{3}e^{3t}.$$

□

17 Second-Order Linear Non-Homogeneous ODE's: The Method of Variation of Parameters

We have seen that solving a second-order linear ODE reduces to finding a general solution to the corresponding homogeneous equation, which is usually easy, together with finding a single particular solution. The method of undetermined coefficients in the previous section is excellent for its simplicity, but it only works in very special cases. We will now give a more general strategy for solving ODE's, which involves recalling the following fact about 2×2 matrices.

Fact 17.1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. Then A is invertible if and only if $\det A \neq 0$, and

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem 17.2 (Variation of Parameters for Second-Order Linear ODE's). Consider the ODE $y'' + p(t)y' + q(t)y = f(t)$, where p , q , and f are continuous. Let $\{y_1, y_2\}$ be a fundamental set of solutions to the corresponding homogeneous equation. Then there exist functions $u_1(t)$, $u_2(t)$ such that

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

is a particular solution to the ODE.

Proof. The strategy of this proof is to approach the equation as a 2-dimensional linear algebra problem. One beautiful feature of this proof is that the Wronskian falls out naturally from the necessary computations.

We work backwards: Suppose there is such a solution $y = u_1y_1 + u_2y_2$. Compute the first derivative:

$$y' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2.$$

At this moment let us make an additional assumption, which we enshrine in a box:

$$\boxed{u_1'y_1 + u_2'y_2 = 0}$$

If we make the above assumption, the derivative simplifies to

$$y' = u_1y_1' + u_2y_2'.$$

Now compute the second derivative:

$$y'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'.$$

Plug everything into the ODE:

$$u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2' + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = f.$$

Regrouping terms yields:

$$u_1[y_1'' + py_1' + qy_1] + u_2[y_2'' + py_2' + qy_2] + u_1'y_1' + u_2'y_2' = f.$$

Since y_1 and y_2 are solutions to the corresponding homogeneous equation, the above reduces to the following boxed equation:

$$\boxed{u_1'y_1' + u_2'y_2' = f}$$

With these computations completed, the question “Is the theorem true?” reduces to the question “Do there exist differentiable functions u_1 and u_2 which satisfy the two boxed equations above?”, in the sense that if the answer to the latter question is “Yes” then so is the answer to the former question.

Regard u_1' and u_2' as unknowns, and y_1 and y_2 as given- then the two boxed equations give a linear system which is equivalent to the following matrix equation:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

(In the above we have omitted the argument t from each of the functions, but we are implicitly considering the equation above for all possible values of t .)

Let $A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$. Since y_1 and y_2 are linearly independent, the determinant $\det A = W(y_1, y_2)$ is non-zero for all t , and hence the matrix $A(t)$ is invertible for all t . Moreover, we can compute the inverse using the familiar formula for 2×2 -matrix inverses:

$$A^{-1} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}.$$

Multiplying on the left of our matrix equation by A^{-1} on both sides, we get

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix}.$$

In other words the equation has a solution and we get $u_1' = -\frac{y_2 f}{W(y_1, y_2)}$ and $u_2' = \frac{y_1 f}{W(y_1, y_2)}$. The functions u_1 and u_2 may now be obtained through integration:

$$u_1 = \int \frac{-y_2 f}{W(y_1, y_2)} dt \text{ and } u_2 = \int \frac{y_1 f}{W(y_1, y_2)} dt.$$

□

Technique 17.3 (General Method for Solving Second-Order Linear ODE's). Given an ODE of the form $y'' + p(t)y' + q(t)y = f(t)$:

- (1) Find a fundamental set of solutions $\{y_1, y_2\}$ to the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$.
- (2) Compute the Wronskian $W(y_1, y_2)$.
- (3) Set $u_1'(t) = -\frac{y_2(t)f(t)}{W(y_1, y_2)(t)}$ and $u_2'(t) = \frac{y_1(t)f(t)}{W(y_1, y_2)(t)}$, and integrate to find u_1 and u_2 .

(4) A particular solution to the ODE is given by $y_p = u_1y_1 + u_2y_2$, where u_1 and u_2 are chosen as above.

(5) The general solution to the ODE is $y = y_p + C_1y_1 + C_2y_2$.

Example 17.4. Solve $y'' - 2y' + y = e^t \ln t$, $t > 0$.

Example 17.5. Solve $y'' + 4y = \sec(2t)$, $y(0) = 1$, $y'(0) = 1$, $-\frac{\pi}{4} < t < \frac{\pi}{4}$.

18 Remarks on Higher-Order Linear ODEs

The student has probably surmised that many of the linear algebra techniques we use to solve second-order ODE's are likely to generalize to higher-order problems. The purpose of this section is to briefly confirm the aforementioned thought. We omit the vast majority of the details due to a lack of sufficient lecture time.

Theorem 18.1 (Existence and Uniqueness). *Consider the n -th order linear ODE $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_2(t)y'' + a_1(t)y'(t) + a_0(t)y = f(t)$, where a_0, a_1, \dots, a_{n-1} , and f are all continuous functions of t . Then there exists one and only one solution to the ODE satisfying the given initial conditions $y'(t_0) = y_0$, $y''(t_0) = y_0', \dots$, $y^{(n-1)}(t_0) = y_0^{(n-1)}$.*

Theorem 18.2 (Principle of Superposition). *Let y_1, \dots, y_n all be solutions to the n -th order linear homogeneous ODE $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_2(t)y'' + a_1(t)y'(t) + a_0(t)y = 0$. Then any function of the form $y = C_1y_1 + C_2y_2 + \dots + C_ny_n$, where C_1, \dots, C_n are constants, is also a solution to the ODE.*

Theorem 18.3 (General Solutions to n -th Order Linear Homogeneous ODE's). *Let y_1, \dots, y_n all be solutions to the n -th order linear homogeneous ODE $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_2(t)y'' + a_1(t)y'(t) + a_0(t)y = 0$, where a_0, a_1, \dots, a_{n-1} are all continuous functions of t . Then all solutions to the ODE are of the form $y = C_1y_1 + C_2y_2 + \dots + C_ny_n$ for some constants C_1, \dots, C_n .*

Solving n -th order linear homogeneous ODE's with constant coefficients is always theoretically possible, thanks to the following well-known theorem (which requires some complex analysis to prove).

Theorem 18.4 (Fundamental Theorem of Algebra). *Every polynomial of degree n has exactly n complex roots (counting multiplicities).*

Example 18.5. Given that $(\lambda - 1)^3(\lambda - 2)(\lambda + i)(\lambda - i) = \lambda^6 - 6\lambda^5 + 13\lambda^4 - 16\lambda^3 + 15\lambda^2 - 10\lambda + 3$, solve $y^{(6)} - 6y^{(5)} + 16y^{(4)} - 16y^{(3)} + 15y'' - 10y' + 3y = 0$.

Solution. The characteristic polynomial has roots 2, i , $-i$, and 1 with a multiplicity of 3. The 2 gives a solution of e^{2t} , while the complex conjugate roots $\pm i$ give solutions of $\cos t$ and $\sin t$. The root 1 gives the solution e^t , while the multiplicity of 3 for this root implies that te^t and t^2e^t are also solutions (a root of multiplicity 3 should give 3 distinct solutions). So a fundamental set of solutions is

$$\{e^{2t}, \cos t, \sin t, e^t, te^t, t^2e^t\}$$

and a general solution to the ODE is given by

$$y = C_1e^{2t} + C_2 \cos t + C_3 \sin t + C_4e^t + C_5te^t + C_6t^2e^t.$$

Note that the solution set to this 6-th order linear homogeneous ODE is a 6-dimensional vector space. This nice correspondence always happens. \square

Example 18.6. Solve $2y^{(6)} - 7y^{(5)} - 4y^{(4)} = 0$.

Theorem 18.7 (DeMoivre's Formula). *For any real numbers x and n , we have*

$$(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx).$$

Proof. This follows from Euler's formula:

$$\begin{aligned}(\cos x + i \sin x)^n &= (e^{ix})^n \\ &= e^{i(nx)} \\ &= \cos(nx) + i \sin(nx).\end{aligned}$$

□

Corollary 18.8 (Roots of Unity). *For every integer n , there are exactly n distinct complex numbers (called the n -th roots of unity) which satisfy the equation $z^n = 1$. These roots, denoted ω_k for $1 \leq k \leq n$, are given by*

$$\omega_k = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right).$$

Note that the ω_k 's form the vertices of a regular n -sided polygon inscribed within the unit circle of the complex plane. Also note that $\omega_k = (\omega_1)^k$ for each k and that $\omega_n = 1$.

Proof. This is an easy consequence of DeMoivre's formula (or Euler's formula), if you note that $e^{2\pi ki} = 1$ for any integer value of k . □

Example 18.9. Solve $y^{(8)} - y = 0$.

Solution. The characteristic equation is $\lambda^8 - 1 = 0$, or $\lambda^8 = 1$. Thus the solutions are the 8-th roots of unity:

$$\lambda = e^{\pi i/4}, e^{2\pi i/4}, e^{3\pi i/4}, e^{4\pi i/4}, e^{5\pi i/4}, e^{6\pi i/4}, e^{7\pi i/4}, e^{8\pi i/4}.$$

Rewriting these solutions as real numbers and complex conjugate pairs:

$$\lambda = 1, -1, \pm i, \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i.$$

The above roots yield the following fundamental set of solutions to the ODE:

$$\{e^t, e^{-t}, \cos t, \sin t, e^{\sqrt{2}t/2} \cos(\frac{\sqrt{2}}{2}t), e^{\sqrt{2}t/2} \sin(\frac{\sqrt{2}}{2}t), e^{-\sqrt{2}t/2} \cos(\frac{\sqrt{2}}{2}t), e^{-\sqrt{2}t/2} \sin(\frac{\sqrt{2}}{2}t)\}.$$

A general solution is any linear combination of the eight functions above. □

Example 18.10. Solve $y^{(5)} + 4y''' = 48t - 6 - 10e^{-t}$.

Solution. We will solve this non-homogeneous equation using the method of undetermined coefficients, i.e., we will guess the form of a solution and then use computations to refine our guess.

We begin by solving the corresponding homogeneous equation $y^{(5)} + 4y''' = 0$. This has characteristic equation $\lambda^5 + 4\lambda^3 = \lambda^3(\lambda^2 + 4) = \lambda^3(\lambda + 2i)(\lambda - 2i) = 0$, whose roots are 0 (with multiplicity 3), $2i$, and $-2i$. So a fundamental set of solutions to the corresponding homogeneous equation is

$$\{1, t, t^2, \cos(2t), \sin(2t)\}.$$

Now we will make an educated guess as to the form of a particular solution. Since we see an e^{-t} on the right-hand side of the ODE, and e^{-t} is not a solution to the corresponding homogeneous equation, we guess that some multiple of e^{-t} will appear in our solution. Since we see a linear function $48t - 6$ on the right-hand side, we are inclined to guess that some linear combination of the functions t and 1 might yield a solution; however, both of these are solutions to the corresponding homogeneous equation. So

we scale up by t^3 (where the exponent 3 is the smallest choice that will prevent our guesses from being solutions to the corresponding homogeneous equation) and guess a linear combination of $t^4 = t^3 \cdot t$ and $t^3 = t^4 \cdot 1$ instead. Thus we assume that we will be able to find a solution of the form

$$y = Ae^{-t} + Bt^4 + Ct^3.$$

Now compute derivatives:

$$\begin{aligned} y' &= -Ae^{-t} + 4Bt^3 + 3Ct^2 \\ y'' &= Ae^{-t} + 12Bt^2 + 6Ct \\ y''' &= -Ae^{-t} + 24Bt + 6C \\ y^{(4)} &= Ae^{-t} + 24B \\ y^{(5)} &= -Ae^{-t} \end{aligned}$$

Plugging the above into the ODE, we get

$$-Ae^{-t} + 4(-Ae^{-t} + 24Bt + 6C) = -5Ae^{-t} + 96Bt + 24C = 48t - 6 - 10e^{-t}.$$

The above implies $-5A = -10$, $96B = 48$, and $24C = -6$, which in turn imply that $A = 2$, $B = \frac{1}{2}$, and $C = -\frac{1}{4}$. This gives us a particular solution to the ODE. And since we already know a fundamental set of solutions to the corresponding homogeneous equation, we obtain a full general solution to the ODE:

$$y = 2e^{-t} + \frac{1}{2}t^4 - \frac{1}{4}t^3 + C_1 + C_2t + C_3t^2 + C_4 \cos(2t) + C_5 \sin(2t).$$

□

Definition 18.11. Let y_1, y_2, \dots, y_n be $(n-1)$ -times differentiable functions, where n is some positive integer. The **Wronskian** of this collection of functions is

$$W(y_1, y_2, \dots, y_n) = \det \begin{bmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ y_1' & y_2' & y_3' & \dots & y_n' \\ y_1'' & y_2'' & y_3'' & \dots & y_n'' \\ \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}.$$

Theorem 18.12. Let y_1, \dots, y_n all be solutions to the n -th order linear homogeneous ODE $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_2(t)y'' + a_1(t)y' + a_0(t)y = 0$, where a_0, a_1, \dots, a_{n-1} are all continuous functions of t . Then y_1, \dots, y_n are linearly dependent if and only if $W(y_1, \dots, y_n) = 0$.

Theorem 18.13 (Variation of Parameters). Consider an n -th order linear ODE $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$. Let $\{y_1, y_2, \dots, y_n\}$ be a fundamental set of solutions to the corresponding homogeneous equation. Let A denote the $n \times n$ matrix defined in the definition of the Wronskian above, i.e., let A be the matrix for which $W(y_1, \dots, y_n) = \det A$. For each integer k with $1 \leq k \leq n$, let A_k be the $(n-1) \times (n-1)$ matrix obtained by omitting the k -th column and the bottom row from A . Let $W_k(y_1, \dots, y_n) = \det A_k$. Then there exist functions $u_1(t), u_2(t), \dots, u_n(t)$ for which

$$y_p = u_1y_1 + u_2y_2 + \dots + u_ny_n$$

is a particular solution to the ODE. Moreover, each u_k ($1 \leq k \leq n$) satisfies the equation

$$u_k' = (-1)^n \frac{W_k(y_1, \dots, y_n) \cdot f}{W(y_1, \dots, y_n)}.$$

Example 18.14. Solve $ty''' + 3y'' = -t^{-3}$, given that a fundamental set of solutions to the corresponding homogeneous equation is $\{1, t, t^{-1}\}$.

19 The Laplace Transform: Definition and Basics

The central idea of the Laplace transform is this: to each function f we permanently associate a new function $F = \mathcal{L}f$ (called its Laplace transform). For “reasonable” functions f , this induces a one-to-one correspondence, i.e. each reasonable f has a Laplace transform and that transform is uniquely determined. The reason for making this association is that calculus operations (taking derivatives, antiderivatives) on a function f end up corresponding to algebraic operations (adding, subtracting, multiplying) on that function’s Laplace transform F . In other words, a “difficult” calculus problem, such as solving a differential equation, may be reduced to an “easy” algebra problem by applying Laplace transforms. This principle is most strongly embodied in the upcoming Theorems 20.5 and 22.1.

Definition 19.1. Let $f(t)$ be a function defined on the interval $(0, \infty)$. The **Laplace transform** of f , denoted $\mathcal{L}f$, is the function (of an independent variable s)

$$\mathcal{L}f(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

provided that the improper integral exists.

As a shorthand notation, we will frequently refer to the Laplace transform of a function with a lower-case letter name, by the capital letter, i.e. for $f(t)$ we will often denote $F(s) = \mathcal{L}f(s)$. We will also sometimes use the longer notation $\mathcal{L}\{f(t)\}(s)$, when we wish to call attention to the respective arguments t and s .

Remark 19.2. The student should recognize that the Laplace transform operator \mathcal{L} is a “function of functions,” i.e., \mathcal{L} takes a function f for input and returns a function $F = \mathcal{L}f$ as output. In this way, it is similar to the operation of taking a derivative, since the derivative operator $\frac{d}{dt}$ takes a function f for input and returns a function $f' = \frac{d}{dt}f$ as output.

However, conceptually there is a big difference between the operators \mathcal{L} and $\frac{d}{dt}$ that we wish to emphasize, which is this: The derivative operator $\frac{d}{dt}$ can be regarded as depending “locally” on f , in the sense that if you fix any particular real input t_0 , the value of the derivative $f'(t_0)$ at t_0 depends only on the values of $f(t)$ for inputs t in a very small neighborhood around t_0 . (Think about the definition and meaning of the derivative.) In other words, if you know what is happening with f “very close” to t_0 then you know the value of $f'(t_0)$.

The Laplace transform $\mathcal{L}f$ is quite different- it depends “globally” on f . If you fix a particular input s_0 and wish to know the value of $\mathcal{L}f(s_0)$, then you are computing the integral of $e^{-s_0 t} f(t)$ over the entire interval $[0, \infty)$, and therefore $\mathcal{L}f(s_0)$ depends not just on the value of f at inputs close to s_0 , but in fact it depends on the value of f at *every* possible input on the right-half line $[0, \infty)$.

This is the reason why, when we go from a function $f(t)$ to its Laplace transform $F(s)$, we choose to change the variable name from t to s - we are emphasizing the fact that any given any real number s , the value of the Laplace transform $F(s)$ depends on the values $f(t)$ for all t in $[0, \infty)$.

Remark 19.3. For most contexts in which the Laplace transform is used, one actually thinks of the map $F = \mathcal{L}f$ as a complex-valued function of a complex variable $F : \mathbb{C} \rightarrow \mathbb{C}$. However, for our purposes in this introductory course, it will suffice to regard F as a real-valued function of a real variable, i.e. $F : \mathbb{R} \rightarrow \mathbb{R}$, and we do so until further notice.

Example 19.4. Compute $\mathcal{L}f(s)$, if possible, for the following functions $f(t)$.

(1) $f(t) = 1$

(2) $f(t) = C$ for constant C

(3) $f(t) = e^{at}$ for constant a

$$(4) f(t) = t$$

$$(5) f(t) = \sin kt$$

$$(6) f(t) = t^t$$

Solution. (a) If $f(t) = 1$, we have

$$\begin{aligned} \mathcal{L}f(s) &= \int_0^{\infty} e^{-st} \cdot 1 dt \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t=b} \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{s} e^{-sb} \right) + \frac{1}{s} \\ &= \frac{1}{s}. \end{aligned}$$

Note that the above computation is only true for $s > 0$; if $s \leq 0$, then the integral in the definition of the Laplace transform diverges.

(b) If $f(t) = C$, then $\mathcal{L}f(s) = \int_0^{\infty} e^{-st} \cdot C dt = C \int_0^{\infty} e^{-st} dt = C \mathcal{L}\{1\}(s) = \frac{C}{s}$, by our computation in part (a).

(c) For $f(t) = e^{at}$, compute:

$$\begin{aligned} \mathcal{L}f(s) &= \int_0^{\infty} e^{-st} e^{at} dt \\ &= \int_0^{\infty} e^{-(s-a)t} dt \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{s-a} e^{-(s-a)t} \right]_{t=0}^{t=b} \\ &= \frac{1}{s-a}. \end{aligned}$$

As in part (a), we note that the computation above is only valid for $s > a$.

(d) For $f(t) = t$, we use integration by parts:

$$\begin{aligned}
\mathcal{L}f(s) &= \int_0^\infty e^{-st} t dt \\
&= \lim_{b \rightarrow \infty} \left[-\frac{t}{s} e^{-st} \right]_{t=0}^{t=b} + \int_0^b \frac{1}{s} e^{-st} dt \\
&= \lim_{b \rightarrow \infty} \left[-\frac{t}{s} e^{-st} \right]_{t=0}^{t=b} + \left[-\frac{1}{s^2} e^{-st} \right]_{t=0}^{t=b} \\
&= 0 + \frac{1}{s^2} \\
&= \frac{1}{s^2}.
\end{aligned}$$

The above holds only for $s > 0$.

(e) For $f(t) = \sin(kt)$, we need to use integration by parts twice. (Students may remember integrating functions of the form $e^{at} \sin(bt)$ as a challenging integration by parts exercise in Cal II.)

$$\begin{aligned}
\mathcal{L}f(s) &= \int_0^\infty e^{-st} \sin(kt) dt \\
&= \lim_{b \rightarrow \infty} \left[-\frac{1}{s} e^{-st} \sin(kt) \right]_{t=0}^{t=b} + \frac{k}{s} \int_0^b e^{-st} \cos(kt) dt \\
&= 0 + \frac{k}{s} \int_0^\infty e^{-st} \cos(kt) dt \\
&= \frac{k}{s} \left[\lim_{a \rightarrow \infty} \left[-\frac{1}{s} e^{-st} \cos(kt) \right]_{t=0}^{t=a} - \frac{k}{s} \int_0^a e^{-st} \sin(kt) dt \right] \\
&= \frac{k}{s} \left[\frac{1}{s} - \frac{k}{s} \int_0^\infty e^{-st} \sin(kt) dt \right] \\
&= \frac{k}{s^2} - \frac{k^2}{s^2} \mathcal{L}f(s).
\end{aligned}$$

Taking the first and last lines above together, we write:

$$\left(1 + \frac{k^2}{s^2}\right) \mathcal{L}f(s) = \frac{k}{s^2}$$

and therefore

$$\mathcal{L}f(s) = \frac{k}{s^2 + k^2}.$$

(f) If $f(t) = t^t$, then the Laplace transform $\mathcal{L}f(s)$ does not exist for any value of s . This is because the definition of the Laplace transform gives $\mathcal{L}f(s) = \int_0^\infty e^{-st} t^t dt$; but t^t grows faster in the long term than e^{st} does for any value of s ; therefore $e^{-st} t^t \rightarrow \infty$. It follows that the indefinite integral $\int_0^\infty e^{-st} t^t dt$ cannot possibly converge for any value of s . \square

Example 19.5. Compute $\mathcal{L}\{\sinh t\}$, where $\sinh t = \frac{1}{2}(e^x - e^{-x})$. (This computation strongly hints that the Laplace transform is linear.)

Fact 19.6 (Laplace Transforms of Common Functions).

$f(t)$	$F(s) = \mathcal{L}f(s)$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$\sin(kt)$	$\frac{k}{s^2 + k^2}$
$\sinh(kt) = \frac{1}{2}(e^{kt} - e^{-kt})$	$\frac{k}{s^2 - k^2}, s > k$
t	$\frac{1}{s^2}, s > 0$

Definition 19.7. A function f of an independent variable t has **exponential order** b , where b is a real number, if there exist constants $C > 0$ and $T > 0$ for which

$$|f(t)| \leq Ce^{bt} \text{ for all } t > T.$$

Theorem 19.8. If $f(t)$ is a piecewise continuous function of exponential order b , then the Laplace transform $\mathcal{L}f(s)$ exists for $s > b$.

We will state the next lemma without proof, but drawing an appropriate picture lends it a lot of credibility.

Lemma 19.9. For any integrable function $g(t)$ on any interval $[a, b]$, $|\int_a^b g(t)dt| \leq \int_a^b |g(t)|dt$.

Theorem 19.10. If $f(t)$ is a piecewise continuous function of exponential order b and $F = \mathcal{L}f$, then $\lim_{s \rightarrow \infty} F(s) = 0$.

Proof. It suffices (by the squeeze theorem) to prove that $\lim_{s \rightarrow \infty} |F(s)| = 0$. Since f is of exponential order b , there are constants C and T such that $|f(t)| \leq Ce^{bt}$ for all $t > T$. Then by the previous lemma, for all $t > T$ we have

$$\begin{aligned} |F(s)| &= \left| \int_0^{\infty} e^{-st} f(t) dt \right| \\ &\leq \int_0^{\infty} |e^{-st} f(t)| dt \\ &= \int_0^{\infty} e^{-st} |f(t)| dt \\ &\leq \int_0^{\infty} e^{-st} C e^{bt} dt \\ &= C \int_0^{\infty} e^{-st} e^{bt} dt \\ &= C \mathcal{L}\{e^{bt}\}(s) \\ &= C \cdot \frac{1}{s-b}. \end{aligned}$$

It follows then that $\lim_{s \rightarrow \infty} |F(s)| \leq \lim_{s \rightarrow \infty} \frac{C}{s-b} = 0$. This completes the proof. \square

20 Properties of the Laplace Transform

Theorem 20.1 (Linearity of the Laplace Transform). *The Laplace transform operator \mathcal{L} is linear. That is,*

$$\mathcal{L}(af + bg) = a\mathcal{L}f + b\mathcal{L}g$$

for any functions f and g and any constants a and b .

Proof. This is just because the integral operator \int_0^∞ is linear as well; plug into the definition and check. \square

Example 20.2. Compute $\mathcal{L}\{5 - 2e^{-t}\}$.

Solution. By the linearity of \mathcal{L} ,

$$\begin{aligned} \mathcal{L}\{5 - 2e^{-t}\}(s) &= 5\mathcal{L}\{1\}(s) - 2\mathcal{L}\{e^{-t}\}(s) \\ &= 5 \cdot \frac{1}{s} - 2 \cdot \frac{1}{s+1} \\ &= \frac{3s+5}{s(s+1)}, \end{aligned}$$

for $s > 0$. \square

Theorem 20.3 (Shifting Property of the Laplace Transform). *Let $f(t)$ be a function and suppose $F(s) = \mathcal{L}f(s)$ exists for $s > b$. Then*

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s - a)$$

for $s > b + a$.

Proof. This is just the definition of the Laplace transform again- plug it in and see. \square

Example 20.4. Compute $\mathcal{L}f$ where

- (1) $f(t) = e^{-2t} \sin t$.
- (2) $f(t) = 4te^{3t}$.

Solution. (a) By the shifting property,

$$\begin{aligned} \mathcal{L}\{e^{-2t} \sin t\}(s) &= \mathcal{L}\{\sin t\}(s+2) \\ &= \frac{1}{(s+2)^2 + 1}. \end{aligned}$$

(b) By the shifting property as well as the linearity property,

$$\begin{aligned} \mathcal{L}\{4te^{3t}\}(s) &= 4\mathcal{L}\{e^{3t}t\}(s) \\ &= 4\mathcal{L}\{t\}(s-3) \\ &= \frac{4}{(s-3)^2}. \end{aligned}$$

\square

Theorem 20.5 (Laplace Transform of a Derivative). *Let $f(t)$ be a differentiable function of exponential order b and let $F = \mathcal{L}f$. Then*

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$$

for $s > b$.

Proof. The proof is integration by parts (taking $u = e^{-st}$ and $dv = f'(t)dt$):

$$\begin{aligned} \mathcal{L}\{f'(t)\}(s) &= \int_0^{\infty} e^{-st} f'(t) dt \\ &= \lim_{a \rightarrow \infty} [e^{-st} f(t)]_{t=0}^{t=a} + \int_0^a s e^{-st} f(t) dt \\ &= \lim_{a \rightarrow \infty} e^{-sa} f(a) - f(0) + s \int_0^a e^{-st} f(t) dt \\ &= 0 - f(0) + \mathcal{L}f(s) \\ &= sF(s) - f(0). \end{aligned}$$

The fact that the third line is equal to the fourth line, i.e. that $e^{-sa} f(a)$ approaches 0 as $a \rightarrow \infty$, is true when $s > b$ because f is of exponential order b . \square

Example 20.6. Compute $\mathcal{L}\{\cos(kt)\}$.

Solution. Note that if $f(t) = \sin(kt)$ then $f'(t) = k \cos(kt)$. So we apply the previous theorem (and also use the linearity of \mathcal{L}):

$$\begin{aligned} \mathcal{L}\{\cos(kt)\}(s) &= \frac{1}{k} \mathcal{L}\{k \cos(kt)\}(s) \\ &= \frac{1}{k} [s \mathcal{L}\{\sin(kt)\}(s) - \sin(k \cdot 0)] \\ &= \frac{1}{k} \cdot s \left(\frac{k}{s^2 + k^2} \right) \\ &= \frac{s}{s^2 + k^2}. \end{aligned}$$

\square

Theorem 20.7 (Derivatives of the Laplace Transform). *If $f(t)$ is a piecewise continuous function of exponential order b , and $F = \mathcal{L}f$ is its Laplace transform, then for $s > b$,*

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s).$$

Proof Sketch. First note that, by an easy computation, $\frac{d^n}{ds^n} e^{-st} = (-1)^n t^n e^{-st}$. Now the proof consists of the following string of equalities which turn out to be true:

$$\begin{aligned}
F^{(n)}(s) &= \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}(s) \\
&= \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t) dt \\
&= \int_0^\infty \left[\frac{d^n}{ds^n} e^{-st} f(t) \right] dt \\
&= \int_0^\infty [(-1)^n t^n e^{-st}] f(t) dt \\
&= (-1)^n \int_0^\infty e^{-st} t^n f(t) dt \\
&= (-1)^n \mathcal{L}\{t^n f(t)\}(s).
\end{aligned}$$

All of the equalities between the lines above should be clear to a differential equations student, except the equality between lines 2 and 3. The fact that the n -th derivative (with respect to s) can be pulled inside the integral (with respect to t) without changing the value of the expression is not obvious, and is in fact not true for all functions of two variables s and t —however, it is true for a special class of functions which includes the function $e^{-st} f(t)$, because $f(t)$ is of exponential order. The student may see a proof of this fact sometime later in a good real analysis course. \square

Example 20.8. Compute the Laplace transforms of

(1) $t \cos(2t)$

(2) $t^2 e^{-3t}$

(3) t^n

Solution. (a) Using the previous theorem,

$$\begin{aligned}
\mathcal{L}\{t \cos(2t)\}(s) &= (-1)^1 \left(\frac{d}{ds} \mathcal{L}\{\cos(2t)\}(s) \right) \\
&= - \left(\frac{d}{ds} \frac{s}{s^2 + 4} \right) \\
&= - \frac{s^2 + 4 - 2s^2}{(s^2 + 4)^2} \\
&= \frac{s^2 - 4}{(s^2 + 4)^2}.
\end{aligned}$$

(b) Again using the previous theorem:

$$\begin{aligned}
\mathcal{L}\{t^2 e^{-3t}\}(s) &= (-1)^2 \left(\frac{d^2}{ds^2} \mathcal{L}\{e^{-3t}\}(s) \right) \\
&= \frac{d^2}{ds^2} \frac{1}{s + 3} \\
&= \frac{2}{(s + 3)^3}.
\end{aligned}$$

(c) Note that $\frac{d^n}{ds^n} \frac{1}{s} = (-1)^n \frac{n!}{s^{n+1}}$. Then:

$$\begin{aligned} \mathcal{L}\{t^n\}(s) &= (-1)^n \left(\frac{d^n}{ds^n} \mathcal{L}\{1\}(s) \right) \\ &= (-1)^n \left(\frac{d^n}{ds^n} \frac{1}{s} \right) \\ &= (-1)^n \cdot (-1)^n \frac{n!}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}}. \end{aligned}$$

□

Example 20.9. Compute the Laplace transforms of f , f' , and f'' for $f(t) = (3t - 1)^3$.

Fact 20.10 (More Laplace Transforms of Common Functions).

$f(t)$	$F(s) = \mathcal{L}f(s)$
t^n	$\frac{n!}{s^{n+1}}, s > 0$
$\cos(kt)$	$\frac{s}{s^2 + k^2}$
$\cosh(kt) = \frac{1}{2}(e^{kt} - e^{-kt})$	$\frac{s}{s^2 - k^2}, s > k$
$e^{at} \sin(kt)$	$\frac{k}{(s-a)^2 + k^2}$
$e^{at} \cos(kt)$	$\frac{s-a}{(s-a)^2 + k^2}, s > 0$
$e^{at} \sinh(kt)$	$\frac{k}{(s-a)^2 - k^2}$
$e^{at} \cosh(kt)$	$\frac{s-a}{(s-a)^2 - k^2}, s > 0$

21 The Inverse Laplace Transform

Definition 21.1. The **inverse Laplace transform** of a function $F(s)$ is the function $f(t)$ for which $\mathcal{L}f = F$, if such a function exists. If such a function exists, we denote it by $f = \mathcal{L}^{-1}F$.

Example 21.2. Observe that the functions $F(s) = 1$, $F(s) = s$, $F(s) = s^2$, for example, have no inverse Laplace transform. (Use Theorem 19.10.)

Example 21.3. Find the inverse Laplace transform of

$$(1) F(s) = \frac{1}{s-6}$$

$$(2) F(s) = \frac{2}{s^2+4}$$

$$(3) F(s) = \frac{6}{s^4}$$

$$(4) F(s) = \frac{6}{(s+2)^4}$$

Theorem 21.4 (Linearity of the Inverse Laplace Transform). *The inverse Laplace transform operator \mathcal{L}^{-1} is linear. That is,*

$$\mathcal{L}^{-1}(aF + bG) = a\mathcal{L}^{-1}F + b\mathcal{L}^{-1}G$$

for any functions F and G and any constants a and b .

Example 21.5. Find the inverse Laplace transform of

$$(1) \frac{1}{s^3}$$

$$(2) -\frac{7}{s^2+16}$$

$$(3) \frac{1}{s^3} - \frac{7}{s^2+16}$$

$$(4) \frac{5}{s} - \frac{2}{s-10}$$

Example 21.6. Find the inverse Laplace transform of $F(s) = \frac{2s-9}{s^2+25}$.

Solution. We use the linearity of \mathcal{L}^{-1} .

$$\begin{aligned} \mathcal{L}^{-1}F(t) &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}(t) - 9\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\}(t) \\ &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}(t) - \frac{9}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}(t) \\ &= 2\cos(5t) - \frac{9}{5}\sin(5t). \end{aligned}$$

□

Example 21.7. Compute $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\}$. (*Hint:* Rewrite the denominator in the form $(s-a)^2+k^2$.)

22 Solving IVPs with the Laplace Transform

Theorem 22.1 (Laplace Transform of a Higher Derivative). *Let f be an n -times differentiable function of exponential order b , and let $F = \mathcal{L}f$. Then*

$$\mathcal{L}(f^{(n)})(s) = s^n F(s) - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example 22.2. Compute $\mathcal{L}\{\sin^2(kt)\}$.

Solution. If $f(t) = \sin^2(kt)$, then $f'(t) = 2k \sin(kt) \cos(kt) = k \sin(2kt)$. Then it follows from Theorem 22.1 that

$$\mathcal{L}\{k \sin(2kt)\}(s) = s\mathcal{L}\{\sin^2(kt)\}(s) - \sin(2k(0)).$$

So:

$$\begin{aligned} \mathcal{L}\{\sin^2(kt)\}(s) &= \frac{1}{s}\mathcal{L}\{k \sin(2kt)\}(s) \\ &= \frac{k}{s}\mathcal{L}\{\sin(2kt)\}(s) \\ &= \frac{k}{s} \cdot \frac{2k}{s^2+4k^2} \\ &= \frac{2k^2}{s(s^2+4k^2)}. \end{aligned}$$

□

Example 22.3. Solve the IVP $y' - 4y = e^{4t}$, $y(0) = 0$.

Solution. Let $Y = \mathcal{L}y$. Take the Laplace transform of both sides of the ODE:

$$\mathcal{L}(y' - 4y) = \mathcal{L}\{e^{4t}\}$$

By the linearity of \mathcal{L} and by Theorem 22.1, the above translates into the following equation:

$$(sY - y(0)) - 4Y = \frac{1}{s-4}$$

After simplifying and grouping like terms this gives:

$$(s-4)Y = \frac{1}{s-4}$$

It follows that $\mathcal{L}y = Y = \frac{1}{(s-4)^2}$. Then

$$y = \mathcal{L}^{-1}Y = \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = te^{4t}$$

and the problem is finished. (Check that this answer is actually a solution to the IVP.) □

Example 22.4. Solve $y'' - 4y' = 0$, $y(0) = 3$, $y'(0) = 8$.

Solution. Again let $Y = \mathcal{L}y$, and take the Laplace transform of both sides of the ODE:

$$(s^2Y - sy(0) - y'(0)) - 4(sY - y(0)) = 0$$

Plugging in initial conditions and grouping like terms, we get:

$$(s^2 - 4s)Y - 3s + 4 = 0$$

and hence $Y = \frac{3s-4}{s^2-4s} = \frac{3s-4}{s(s-4)}$. Now one can use method of partial fractions to show that $Y = \frac{1}{s} + \frac{2}{s-4}$. To find y , we compute the inverse Laplace transform of Y :

$$\begin{aligned} y &= \mathcal{L}^{-1}Y \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s-4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} \\ &= 1 + 2e^{4t}. \end{aligned}$$

This completes the problem. □

Example 22.5. Solve $y'' + 2y' + y = 6$, $y(0) = 5$, $y'(0) = 10$.

Example 22.6. Solve $y''' + 4y' = -10e^t$, $y(0) = 2$, $y'(0) = 2$, $y''(0) = -10$.

Example 22.7. Solve $y'' + 3ty' - 6y = 3$, $y(0) = y'(0) = 0$.

Solution. Note that this example is interesting because it is a second-order ODE where the coefficients are not constant. So the methods we developed earlier in the course for second-order ODE's are not obviously useful. But the Laplace transform turns out to be adequate to the task.

As usual, we let $Y = \mathcal{L}y$ and take Laplace transforms of both sides of the ODE. The tricky thing here is computing $\mathcal{L}\{3ty'\}$. By Theorem 20.7, we have

$$\begin{aligned}\mathcal{L}\{3ty'\} &= -\frac{d}{ds}\mathcal{L}\{3y'\} \\ &= -3 \cdot \frac{d}{ds}(sY - y(0)) \\ &= -3 \cdot \frac{d}{ds}(sY) \\ &= -3(Y + sY').\end{aligned}$$

Then, applying \mathcal{L} to both sides of the ODE, we get:

$$s^2Y - sy(0) - y'(0) - 3(Y + sY') - 6Y = \frac{3}{s}.$$

Setting $y(0) = y'(0) = 0$ and rearranging terms, we get

$$-3sY' + (s^2 - 9)Y = \frac{3}{s}.$$

Thus we have reduced the problem to a first-order linear ODE, which we can solve using an integrating factor. Rewrite the ODE in standard form as follows:

$$Y' + \left(-\frac{1}{3}s + 3s^{-1}\right)Y = -\frac{1}{s^2}.$$

Set $\mu(s) = e^{\int(-1/3)s+3s^{-1})ds} = e^{-(1/6)s^2+3\ln s}$, and multiply on both sides of the equation by μ . This gives us

$$\begin{aligned}\frac{d}{ds}(e^{-(1/6)s^2+3\ln s}Y) &= -\frac{1}{s^2}e^{-(1/6)s^2+3\ln s} \\ &= -\frac{1}{s^2}e^{-(1/6)s^2} \cdot s^3 \\ &= -se^{-(1/6)s^2}.\end{aligned}$$

It follows that

$$e^{-(1/6)s^2+3\ln s}Y = \int(-se^{(1/6)s^2})ds = 3e^{-(1/6)s^2} + C$$

and hence

$$Y = (3e^{-(1/6)s^2} + C) \cdot e^{(1/6)s^2-3\ln s} = \frac{3}{s^3} + \frac{C}{s^3}e^{(1/6)s^2}.$$

Now consider the expression above; we have determined that the function $Y = \mathcal{L}y$ is one of an infinite class of functions, which depend on the choice of constant C . But notice that if C is not equal to zero, then $\lim_{s \rightarrow \infty} \left(\frac{3}{s^3} + \frac{C}{s^3} e^{(1/6)s^2} \right)$ is either ∞ or $-\infty$ (depending on the sign of C); this is impossible if Y is the Laplace transform of y , since we must have $\lim_{s \rightarrow \infty} Y = 0$. So we assume $C = 0$ and hence $Y = \frac{3}{s^3}$. It follows then that

$$y = \mathcal{L}^{-1}Y = \mathcal{L}^{-1}\left\{\frac{3}{s^3}\right\} = \frac{3}{2}t^2.$$

This completes the problem. The reader should verify that y is actually a solution to the IVP. \square

23 Laplace Transforms of Step Functions

Definition 23.1. The **Heaviside function**, or **unit step function**, denoted $\mathcal{U}(t)$, is the piecewise continuous function defined as follows:

$$\mathcal{U}(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t \geq 0. \end{cases}$$

We will frequently work with horizontal shifts of the Heaviside function, i.e. functions of the form $\mathcal{U}(t - a)$ where a is a real number. Note that

$$\mathcal{U}(t - a) = \begin{cases} 0 & \text{if } t < a; \\ 1 & \text{if } t \geq a. \end{cases}$$

We will also find it convenient to work with sums of the form $\mathcal{U}(t - a) - \mathcal{U}(t - b)$, where $a < b$. Note that

$$\mathcal{U}(t - a) - \mathcal{U}(t - b) = \begin{cases} 0 & \text{if } t < a; \\ 1 & \text{if } a \leq t < b; \\ 0 & \text{if } t \geq b. \end{cases}$$

Example 23.2. Sketch graphs of the following functions.

- (1) $\mathcal{U}(t - 5)$
- (2) $\mathcal{U}(t - 3\pi) \sin t$
- (3) $[\mathcal{U}(t - \pi) - \mathcal{U}(t - 3\pi)] \cos t$
- (4) $[\mathcal{U}(t - \pi) - \mathcal{U}(t - 3\pi)] \cos t + \mathcal{U}(t - 3\pi)e^{t-3\pi}$

Theorem 23.3. Suppose $f(t)$ is a function and $F(s) = \mathcal{L}f(s)$ exists for $s > b$. If $a > 0$, then

$$\mathcal{L}\{f(t - a)\mathcal{U}(t - a)\}(s) = e^{-as}F(s)$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\mathcal{U}(t - a).$$

Proof. To prove the first equality, we compute:

$$\begin{aligned}\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} &= \int_0^{\infty} e^{-st} f(t-a)\mathcal{U}(t-a) dt \\ &= \int_0^a e^{-st} f(t-a)\mathcal{U}(t-a) dt + \int_a^{\infty} e^{-st} f(t-a)\mathcal{U}(t-a) dt \\ &= \int_0^a e^{-st} f(t-a) \cdot 0 dt + \int_a^{\infty} e^{-st} f(t-a) \cdot 1 dt \\ &= \int_a^{\infty} e^{-st} f(t-a) dt.\end{aligned}$$

Now use substitution rule with $u = t - a$:

$$\begin{aligned}\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} &= \int_a^{\infty} e^{-st} f(t-a) dt \\ &= \int_0^{\infty} e^{-s(u+a)} f(u) du \\ &= e^{-as} \int_0^{\infty} e^{-su} f(u) du \\ &= e^{-as} F(s).\end{aligned}$$

The second equality in the theorem is just a re-writing of the first equality. □

Example 23.4. Find

- (1) $\mathcal{L}\{\mathcal{U}(t-a)\}$
- (2) $\mathcal{L}\{(t-3)^5\mathcal{U}(t-3)\}$
- (3) $\mathcal{L}\{\sin(t - \frac{\pi}{6})\mathcal{U}(t - \frac{\pi}{6})\}$
- (4) $\mathcal{L}\{t^2\mathcal{U}(t-1)\}$
- (5) $\mathcal{L}\{\mathcal{U}(t-\pi)\sin t\}$

Example 23.5. Find

- (1) $\mathcal{L}^{-1}\{\frac{e^{-4s}}{s^3}\}$
- (2) $\mathcal{L}^{-1}\{\frac{e^{-\pi s/2}}{s^2+16}\}$

Example 23.6. Solve

$$y'' + 9y = \begin{cases} 1 & \text{if } 0 < t < \pi; \\ 0 & \text{if } t \geq \pi; \end{cases}$$

where $y(0) = y'(0) = 0$.

Solution. Let $f(t)$ denote the piecewise continuous function on the right-hand side of the ODE. Rewrite f as follows:

$$f(t) = 1 \cdot [\mathcal{U}(t) - \mathcal{U}(t - \pi)] = 1 - \mathcal{U}(t - \pi).$$

Now we compute the Laplace transform of f according to Theorem 23.3:

$$\mathcal{L}f(s) = \mathcal{L}\{1 - \mathcal{U}(t - \pi)\} = \mathcal{L}\{1\} - \mathcal{L}\{\mathcal{U}(t - \pi)\} = \frac{1}{s} - e^{\pi s} \cdot \frac{1}{s}.$$

Now we apply \mathcal{L} to both sides of the ODE. As usual we denote $Y = \mathcal{L}y$:

$$s^2Y - sy(0) - y'(0) + 9Y = \frac{1}{s} - e^{\pi s} \cdot \frac{1}{s}.$$

By applying the initial conditions $y(0) = y'(0) = 0$ and collecting like terms, the above reduces to:

$$(s^2 + 9)Y = \frac{1}{s}(1 - e^{-\pi s})$$

or

$$Y = \frac{1}{s(s^2+9)}(1 - e^{-\pi s}) = \frac{1}{s(s^2+9)} - \frac{e^{-\pi s}}{s(s^2+9)}.$$

Now to finish the problem it suffices to compute \mathcal{L}^{-1} of the above, since $y = \mathcal{L}^{-1}Y$. Let us do the two terms on the right-hand separately. (This is ok because \mathcal{L}^{-1} is linear.)

The method of partial fractions reveals that $\frac{1}{s(s^2+9)} = \frac{1}{9s} - \frac{s}{9(s^2+9)}$, and hence

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} &= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= \frac{1}{9} - \frac{1}{9}\cos(3t). \end{aligned}$$

Applying Theorem 23.3 again, we have

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+9)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}(t - \pi) \cdot \mathcal{U}(t - \pi) \\ &= \left(\frac{1}{9} - \frac{1}{9}\cos(3(t - \pi))\right)\mathcal{U}(t - \pi). \end{aligned}$$

Combining all of the above computations, we get

$$y = \mathcal{L}^{-1}Y = \frac{1}{9} - \frac{1}{9}\cos(3t) - \left(\frac{1}{9} - \frac{1}{9}\cos(3(t - \pi))\right)\mathcal{U}(t - \pi).$$

The above solution to the IVP may be written as a piecewise function as follows:

$$y = \begin{cases} \frac{1}{9} - \frac{1}{9}\cos(3t) & \text{if } 0 < t < \pi; \\ -\frac{2}{9}\cos(3t) & \text{if } t \geq \pi. \end{cases}$$

□

24 The Convolution Theorem

Definition 24.1. Let $f(t)$ and $g(t)$ be functions with domain $[0, \infty)$. The **convolution** of f and g , denoted $f * g$, is the function (of independent variable t) defined as follows:

$$(f * g)(t) = \int_0^t f(t-u)g(u)du,$$

whenever the integral exists.

Example 24.2. Let $f(t) = t$ and $g(t) = \sin t$ for $t \geq 0$. Compute $f * g$.

Recall the following immensely important theorem from Calculus III:

Lemma 24.3 (Fubini's Theorem). *Let $F(t, y)$ be an integrable function on the rectangular region $R = [a, b] \times [c, d]$. Then f is integrable, and*

$$\int \int_R F(t, y) d(t, y) = \int_c^d \int_a^b F(t, y) dt dy = \int_a^b \int_c^d F(t, y) dy dt.$$

The student should recall that Fubini's theorem is actually stronger than what is stated above, since it is true not just for rectangular regions of integration but actually for much more general regions. We will not benefit much at the moment from attempting to formulate a precise version of this statement, but we will use the stronger version of Fubini's theorem to prove the next fact.

Theorem 24.4 (Convolution Theorem). *Suppose $f(t)$ and $g(t)$ are both piecewise continuous functions of exponential order b . Let $F = \mathcal{L}f$ and $G = \mathcal{L}g$. Then*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g.$$

Proof. Write the product $F(s)G(s)$ using the definition of the Laplace transform:

$$F(s)G(s) = \int_0^\infty e^{-sx} f(x) dx \cdot \int_0^\infty e^{-sy} g(y) dy.$$

The above may be rewritten as an iterated integral:

$$F(s)G(s) = \int_0^\infty \int_0^\infty e^{-s(x+y)} f(x)g(y) dx dy.$$

Now briefly consider the inner integral $\int_0^\infty e^{-s(x+y)} f(x)g(y) dx$ in the equation above (imagining y held constant). Applying the substitution rule with $t = x + y$, we get:

$$\int_0^\infty e^{-s(x+y)} f(x)g(y) dx = \int_y^\infty e^{-st} f(t-y)g(y) dx$$

and therefore

$$F(s)G(s) = \int_0^\infty \int_y^\infty e^{-st} f(t-y)g(y) dt dy.$$

Note that the region of integration in the above is exactly the set $\{(t, y) : 0 \leq t \leq y, 0 \leq y \leq \infty\}$, which corresponds to the "infinite triangle" bounded below by the line $y = 0$ and on the left by the line $y = t$ in the ty -plane. We can rewrite this region of integration as $\{(t, y) : 0 \leq t \leq \infty, 0 \leq y \leq t\}$, and use Fubini's theorem to exchange the order of the iterated integral:

$$\begin{aligned}
F(s)G(s) &= \int_0^\infty \int_0^t e^{-st} f(t-y)g(y)dydt \\
&= \int_0^\infty e^{-st} \left(\int_0^t f(t-y)g(y)dy \right) dt \\
&= \int_0^\infty e^{-st} (f * g)(t) dt \\
&= \mathcal{L}\{(f * g)(t)\}(s).
\end{aligned}$$

This proves the theorem. □

Example 24.5. Verify the convolution theorem for $f(t) = t$ and $g(t) = \sin t$.

Corollary 24.6. *The convolution operation $*$ is commutative. That is,*

$$f * g = g * f$$

for any two functions f and g .

Proof. Although the theorem is true in general (and can be shown via elementary means, e.g. substitution rule), we will just prove the case where both the Laplace transforms $F = \mathcal{L}f$ and $G = \mathcal{L}g$ exist, by using the convolution theorem. Since multiplication is commutative, in this case by the convolution theorem we have:

$$\begin{aligned}
f * g &= \mathcal{L}^{-1}(FG) \\
&= \mathcal{L}^{-1}(GF) \\
&= g * f.
\end{aligned}$$

□

The next corollary is of great theoretical significance. It gives a direct method for producing a solution to any linear ODE with constant coefficients, provided the function $f(t)$ on the right-hand side is Laplace transformable. By careful analysis of characteristic polynomials, one may use Duhamel's principle to deduce the method of variation of parameters.

Corollary 24.7 (Duhamel's Principle). *Let a_0, a_1, \dots, a_n be constants, and let $f(t)$ be a function of exponential order b for some b . Consider the linear IVP:*

$$\begin{aligned}
a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y &= f(t); \\
y(0) = y'(0) = \dots = y^{(n-1)}(0) &= 0.
\end{aligned}$$

Let $p(s)$ be the characteristic polynomial of the above ODE, i.e.

$$p(s) = a_n s^n + \dots + a_2 s^2 + a_1 s + a_0.$$

Then the unique solution to the IVP is given by

$$y = \mathcal{L}^{-1} \left\{ \frac{1}{p(s)} \right\} * f.$$

Proof. Let y be the unique solution to the IVP and let $Y = \mathcal{L}y$. Taking Laplace transforms of both sides of the ODE and plugging in initial conditions yields:

$$(a_n s^n + \dots + a_2 s^2 + a_1 s + a_0)Y = p(s)Y = \mathcal{L}f.$$

Therefore $Y = \frac{1}{p(s)} \cdot \mathcal{L}f$, and hence by the convolution theorem,

$$\begin{aligned} y &= \mathcal{L}^{-1}Y \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{p(s)} \cdot \mathcal{L}f \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{p(s)} \right\} * \mathcal{L}(\mathcal{L}^{-1}f) \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{p(s)} \right\} * f. \end{aligned}$$

□

25 The Gamma Function

Definition 25.1. The **Gamma function**, denoted $\Gamma(x)$, is a complex-valued function of a complex number x with $\operatorname{Re}(x) > 0$, defined as follows:

$$\Gamma(x) = \int_0^{\infty} e^{-u} u^{x-1} du.$$

Example 25.2. Verify mentally that the improper integral ought to converge if x is a real number strictly bigger than 0.

Theorem 25.3. For every complex number x with $\operatorname{Re}(x) > 0$,

$$\Gamma(x+1) = x\Gamma(x).$$

Proof. We use integration by parts on the definition of the Gamma function at $x+1$:

$$\begin{aligned} \Gamma(x+1) &= \int_0^{\infty} e^{-u} u^x du \\ &= [-e^{-u} u^x]_{u=0}^{\infty} + \int_0^{\infty} e^{-u} (x u^{x-1}) du \\ &= 0 + x \int_0^{\infty} e^{-u} u^{x-1} du \\ &= x\Gamma(x). \end{aligned}$$

□

Corollary 25.4. The function $\Gamma(x+1)$ is an extension of the factorial function to complex inputs x with $\operatorname{Re}(x) > -1$, i.e. for every nonnegative integer n ,

$$\Gamma(n+1) = n!.$$

Moreover, the Gamma function is continuous, in fact infinitely differentiable, in fact infinitely complex-differentiable.

Proof. The claim in the last sentence of the corollary is a bit beyond our scope to verify at the moment. However, we have done enough to verify that Γ extends the factorial function. First check it for $n=0$ by using the definition of the Gamma function:

$$\begin{aligned}
\Gamma(0+1) &= \int_0^{\infty} e^{-u} u^0 du \\
&= \int_0^{\infty} e^{-u} du \\
&= [-e^{-u}]_{u=0}^{\infty} \\
&= 0 - (-1) \\
&= 1 \\
&= 0!
\end{aligned}$$

Now use the functional equation given by Theorem 25.3 to check it for the remaining integers n :

$$\begin{aligned}
\Gamma(1+1) &= 1 \cdot \Gamma(1) = 1 \cdot 1 = 1 = 1! \\
\Gamma(2+1) &= 2 \cdot \Gamma(2) = 2 \cdot 1 = 2 = 2! \\
\Gamma(3+1) &= 3 \cdot \Gamma(3) = 3 \cdot 2 = 6 = 3! \\
\Gamma(4+1) &= 4 \cdot \Gamma(4) = 4 \cdot 6 = 24 = 4! \\
\Gamma(5+1) &= 5 \cdot \Gamma(5) = 5 \cdot 24 = 120 = 5! \\
&\dots
\end{aligned}$$

In general if $\Gamma(n+1) = n!$, then $\Gamma((n+1)+1) = (n+1)\Gamma(n+1) = (n+1) \cdot n! = (n+1)!$, so the pattern continues forever. \square

It turns out that the function Γ can actually be extended even further in a natural way to all complex inputs x which are not equal to a nonnegative integer $0, -1, -2, -3, \dots$, and in such a way that it still satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$. One may check that the integral in the definition of the Gamma function does not converge for $x = 0$, i.e. $\Gamma(0)$ is not defined, and therefore there is no way to extend Γ to any negative integer in such a way that $\Gamma(x+1) = x\Gamma(x)$ for all x . (Why?)

Since the Gamma function extends the factorial function, the next theorem is an extension of the rule $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ which we computed earlier.

Theorem 25.5. *Let a be a positive real number. Then*

$$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}.$$

Proof. The trick here is simply to look at the definition of the Laplace transform of t^a , and use substitution rule with $u = st$, which gives $t = s^{-1}u$ and $dt = s^{-1}du$. We proceed:

$$\begin{aligned}
\mathcal{L}\{t^a\}(s) &= \int_0^{\infty} e^{-st} t^a dt \\
&= \int_0^{\infty} e^{-u} (s^{-1}u)^a \cdot s^{-1} du \\
&= s^{-a} \cdot s^{-1} \cdot \int_0^{\infty} e^{-u} u^a du \\
&= \frac{\Gamma(a+1)}{s^{a+1}}.
\end{aligned}$$

□

The next theorem is a very useful computational tool about the Gamma function (and is also surprising, beautiful, and strange). Its usual proof relies on a famous theorem called Weierstrass' factorization theorem, which is beyond our scope.

Theorem 25.6 (Euler's Reflection Formula). *For complex numbers x which are not integers,*

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}.$$

Example 25.7. Use Euler's reflection formula to compute:

- (1) $\Gamma(\frac{1}{2})$
- (2) $\Gamma(-\frac{1}{2})$

Solution. (1) For $x = \frac{1}{2}$, we have

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right)^2 &= \Gamma\left(1 - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\ &= \frac{\pi}{\sin(\pi/2)} \\ &= \pi \end{aligned}$$

and therefore since $\Gamma(\frac{1}{2})$ must be positive, we have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(2) As an intermediate step, note that by Theorem 25.3 we have

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Now using the reflection formula,

$$\begin{aligned} \Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(1 - \frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right) \\ &= \frac{\pi}{\sin(3\pi/2)} \\ &= -\pi \end{aligned}$$

and therefore $\Gamma(-\frac{1}{2}) = -\pi/\Gamma(\frac{3}{2}) = -\pi \cdot \frac{2}{\sqrt{\pi}} = -2\sqrt{\pi}$. □

Example 25.8. Compute:

- (1) $\mathcal{L}\{\sqrt{t}\}$
- (2) $\mathcal{L}\{\frac{1}{\sqrt{t}}\}$

Solution. Using the results of the previous example, we get:

$$(1) \mathcal{L}\{t^{1/2}\} = \frac{\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}, \text{ and}$$

$$(2) \mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \sqrt{\frac{\pi}{s}}.$$

□

26 Systems of First-Order Linear Homogeneous Equations

Definition 26.1. A system of n first-order linear homogeneous ODE's in n functions is a set of equations of the form

$$\begin{aligned}y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \dots + a_{1n}(t)y_n; \\y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \dots + a_{2n}(t)y_n; \\y_3' &= a_{31}(t)y_1 + a_{32}(t)y_2 + \dots + a_{3n}(t)y_n; \\&\dots \\y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \dots + a_{nn}(t)y_n;\end{aligned}$$

where each a_{ij} ($1 \leq i, j \leq n$) is a function of an independent variable t , and each variable y_1, \dots, y_n represents a differentiable function of t .

Example 26.2. Rewrite the following ODE's as systems of first-order ODE's.

$$(1) \quad y'' - 5y' + 6y = 0$$

$$(2) \quad y^{(4)} + 6y^{(3)} - 2y'' - y' + 17y = 0$$

Definition 26.3. A **vector-valued function**, or \mathbb{R}^n -valued function, is a function \vec{y} which takes a real number t for input and returns a vector $\vec{y}(t)$ in \mathbb{R}^n for output. We will typically denote vector-valued functions using column vector notation, so we will write

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dots \\ y_n(t) \end{bmatrix}$$

where each y_1, \dots, y_n is a real-valued function.

If $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$ is a vector valued function and each y_1, \dots, y_n is differentiable, then we define the **derivative** \vec{y}' of \vec{y} to be

$$\vec{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \dots \\ y_n' \end{bmatrix}.$$

A **linear homogeneous ODE of a vector-valued function** is an equation of the form

$$\vec{y}' = A\vec{y},$$

where

$$A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}$$

is an $n \times n$ matrix of functions $a_{ij}(t)$, $1 \leq i, j \leq n$.

Fact 26.4. Every system of n linear homogeneous ODE's in y_1, \dots, y_n (with constant coefficients) cor-

responds uniquely to a linear homogeneous ODE $\vec{y}' = A\vec{y}$ of the vector-valued function $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$

(where A is a matrix with real entries).

Example 26.5. Solve the system

$$\begin{aligned} y_1' &= -y_1 \\ y_2' &= 2y_2 \end{aligned}$$

Solution. It is easy to see that the general solution to the system above is $y_1 = C_1 e^{-t}$, $y_2 = C_2 e^{2t}$, where C_1, C_2 are arbitrary constants.

Note that in this problem, y_1' depends only on y_1 and y_2' depends only on y_2 . This system corresponds to the matrix equation

$$\vec{y}' = A\vec{y}, \text{ where } A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

where A is a diagonal matrix. □

Definition 26.6. A system of n linear homogeneous ODE's is called **decoupled** if each equation in the system is of the form $y_i' = a_{ii}(t)y_i$. Decoupled systems correspond to matrix equations $\vec{y}' = D\vec{y}$ with diagonal matrices D .

We can solve many decoupled systems in one swift stroke.

Theorem 26.7. Let $\vec{y}' = D\vec{y}$ be the matrix representation of a decoupled system of first-order linear ODE's with constant coefficients. (In other words D is a diagonal matrix with real entries.) Let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote the entries along the diagonal of D , and assume that the entries are distinct real numbers. For each $1 \leq i \leq n$, let \vec{e}_i denote the i -th standard basis vector of \mathbb{R}^n , i.e. \vec{e}_i has a 1 for the i -th entry any all 0's in the other entries. Then any vector-valued function of the form

$$\vec{y} = C_1 \vec{e}_1 e^{\lambda_1 t} + C_2 \vec{e}_2 e^{\lambda_2 t} + \dots + C_n \vec{e}_n e^{\lambda_n t}$$

is a solution to the ODE.

Proof. For each $1 \leq i \leq n$, the i -th equation in the system is $y_i' = \lambda_i y_i$, which has general solution $y_i = C_i e^{\lambda_i t}$. So a general solution to the vector-valued ODE is

$$\vec{y} = \begin{bmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \dots \\ C_n e^{\lambda_n t} \end{bmatrix}.$$

The above is just the statement of the theorem written differently. □

27 The Structure of Solutions to Systems of Linear Homogeneous Equations

From this point on we will permanently identify a given system of linear first-order homogeneous ODE's with its associated vector-valued ODE $\vec{y}' = A\vec{y}$, as in the previous section.

Theorem 27.1 (Existence and Uniqueness of Solutions to Systems of Linear First-Order ODE's). *Let $A = A(t)$ be an $n \times n$ matrix whose entries $a_{ij}(t)$ are functions of an independent variable t , and are continuous on an interval I containing t_0 , for $1 \leq i, j \leq n$. Let \vec{y}_0 be any vector in \mathbb{R}^n . Consider the IVP*

$$\vec{y}' = A\vec{y}, \vec{y}(t_0) = \vec{y}_0.$$

Then there is one and only one solution to the IVP on I .

Theorem 27.2. *Let $\vec{y}' = A\vec{y}$ be a system of n linear first-order homogeneous ODE's, where the entries of A are continuous functions of an independent variable t , and let \mathcal{S} be the set of solutions to the ODE. Then \mathcal{S} is an n -dimensional vector space.*

Proof. Let V denote the set of all differentiable real-valued functions of t . The student may easily verify that V together with its usual addition and scalar multiplication is a vector space, i.e., V satisfies all the axioms given in the usual definition of a vector space.

We wish to show \mathcal{S} is a vector space. Since \mathcal{S} is a subset of V , it suffices to check that it is closed under linear combinations. So let \vec{y}_1 and \vec{y}_2 be in \mathcal{S} , i.e. let \vec{y}_1 and \vec{y}_2 be two solutions to the given ODE, and let C_1 and C_2 be constants. Let $\vec{x} = C_1\vec{y}_1 + C_2\vec{y}_2$; we wish to check that \vec{x} is still in \mathcal{S} . This is immediate, since

$$A\vec{x} = A(C_1\vec{y}_1 + C_2\vec{y}_2) = C_1A\vec{y}_1 + C_2A\vec{y}_2 = C_1\vec{y}_1' + C_2\vec{y}_2' = \vec{x}',$$

thus \vec{x} is a solution to the ODE. So \mathcal{S} is indeed a vector space.

We still need to verify that \mathcal{S} is n -dimensional. To show this, we will check that \mathcal{S} is vector-space isomorphic to \mathbb{R}^n . Define a map T from \mathcal{S} to \mathbb{R}^n by the rule

$$T(\vec{y}) = \vec{y}(0).$$

Note that if C_1 and C_2 are constants and \vec{y}_1 and \vec{y}_2 are in \mathcal{S} , then

$$T(C_1\vec{y}_1 + C_2\vec{y}_2) = C_1\vec{y}_1(0) + C_2\vec{y}_2(0) = C_1T(\vec{y}_1) + C_2T(\vec{y}_2),$$

so T is a linear transformation. T is onto by the "existence" part of the Existence and Uniqueness Theorem 27.1: if \vec{y}_0 is any vector in \mathbb{R}^n , then there is a solution \vec{y} in \mathcal{S} with $\vec{y}(0) = T(\vec{y}) = \vec{y}_0$. T is also one-to-one by the "uniqueness" part of Theorem 27.1: if \vec{y}_0 is a vector in \mathbb{R}^n , there is at most one \vec{y} in \mathcal{S} with $\vec{y}(0) = T(\vec{y}) = \vec{y}_0$.

We have shown T is a vector space isomorphism (in a natural way!) of the space of solutions \mathcal{S} with the space of initial conditions \mathbb{R}^n . So \mathcal{S} is n -dimensional as claimed. \square

Corollary 27.3. *Let $\vec{y}' = A\vec{y}$ be a system of n linear first-order homogeneous ODE's, where the entries of A are continuous functions of an independent variable t . Let $\{y_1, y_2, \dots, y_n\}$ be any set of linearly independent solutions to the ODE. Then all solutions to the ODE are of the form $\vec{y} = C_1\vec{y}_1 + \dots + C_n\vec{y}_n$ for some constants C_1, \dots, C_n .*

Proof. This is because any set of n -linearly independent solutions must be a basis for the n -dimensional vector space of all solutions. \square

Definition 27.4. Let $\vec{y}' = A\vec{y}$ be a system of n linear first-order homogeneous ODE's, where the entries of A are continuous functions of an independent variable t . A **fundamental set of solutions** to the ODE is any set $\{y_1, \dots, y_n\}$ of n linearly independent solutions.

28 Solving Systems With Distinct Real Eigenvalues

We need to recall the following definitions and theorems from linear algebra.

Definition 28.1. Let A be an $n \times n$ matrix. A number λ is called an **eigenvalue** of A if there exists a non-zero vector \vec{v} in \mathbb{R}^n for which

$$A\vec{v} = \lambda\vec{v}.$$

Any vector \vec{v} for which $A\vec{v} = \lambda\vec{v}$ is called an **eigenvector** of A corresponding to the eigenvalue λ .

Fact 28.2. A number λ_0 is an eigenvalue of an $n \times n$ matrix A if and only if λ_0 is a root of the characteristic polynomial $\det(A - \lambda I)$.

Fact 28.3. A vector \vec{v} is an eigenvector of A corresponding to the eigenvalue λ if and only if \vec{v} is in the null space of the matrix $A - \lambda I$.

Example 28.4. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & -6 \\ 3 & -7 \end{bmatrix}$.

Solution. The characteristic polynomial is $\det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & -6 \\ 3 & -7 - \lambda \end{bmatrix} = \lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$, so the eigenvalues of A are the roots -5 and 2 .

To find an eigenvector corresponding to the eigenvalue -5 , we are looking for a vector \vec{v} in the null space of $A + 5I = \begin{bmatrix} 9 & -6 \\ 3 & -2 \end{bmatrix}$. This matrix row-reduces to $\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$, which has the same null space. A vector \vec{v} gets sent to 0 by the latter matrix if and only if $3v_1 - 2v_2 = 0$, if and only if $v_1 = \frac{2}{3}v_2$, if and only if \vec{v} is in the span of $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to -5 (and in fact all other eigenvectors are a scalar multiple of it).

A similar computation yields for instance $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (or any scalar multiple of it) as an eigenvector corresponding to 2 . □

Theorem 28.5 (Diagonalizing Matrices with Distinct Eigenvalues). Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding non-zero eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ respectively. Let C be the matrix

$$C = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$$

whose columns are the eigenvectors of A . Then C is invertible, and $A = CDC^{-1}$ and $D = C^{-1}AC$, where D is a diagonal matrix whose entries along the diagonal are $\lambda_1, \dots, \lambda_n$ in order.

Theorem 28.6. Let A be an $n \times n$ matrix with real entries, and suppose that A has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$. Let $\vec{v}_1, \dots, \vec{v}_n$ be eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$ respectively. Then a fundamental set of solutions to the ODE $\vec{y}' = A\vec{y}$ is

$$\{\vec{v}_1 e^{\lambda_1 t}, \dots, \vec{v}_n e^{\lambda_n t}\}.$$

Proof. By Theorem 28.5, $A = CDC^{-1}$ where D is a diagonal matrix whose entries along the diagonal are $\lambda_1, \dots, \lambda_n$ respectively and C is a matrix whose columns are $\vec{v}_1, \dots, \vec{v}_n$ respectively. Then we can rewrite the ODE $\vec{y}' = A\vec{y} = CDC^{-1}\vec{y}$ as

$$C^{-1}\vec{y}' = DC^{-1}\vec{y}.$$

We introduce a change of variable: let $\vec{x} = C^{-1}\vec{y}$. Then $\vec{x}' = C^{-1}\vec{y}'$, since the derivative operator is linear. Plugging this change into the above, we get:

$$\vec{x}' = D\vec{x}.$$

The above represents a decoupled system, and hence by Theorem 26.7, a general solution to it is

$$\vec{x} = C_1\vec{e}_1e^{\lambda_1 t} + \dots + C_n\vec{e}_ne^{\lambda_n t}.$$

Each of the terms in the linear combination on the right-hand side above are obviously linearly independent with respect to one another. So a fundamental set of solutions to $\vec{x}' = D\vec{x}$ is

$$\{\vec{e}_1e^{\lambda_1 t}, \dots, \vec{e}_ne^{\lambda_n t}\}.$$

It follows from our change of variables that for each $1 \leq i \leq n$, the function

$$\begin{aligned} \vec{y} &= C\vec{x} \\ &= C\vec{e}_ie^{\lambda_it} \\ &= \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix} \vec{e}_ie^{\lambda_it} \\ &= \vec{v}_ie^{\lambda_it} \end{aligned}$$

is a solution to $\vec{y}' = A\vec{y}$. Thus the set

$$\{\vec{v}_1e^{\lambda_1 t}, \dots, \vec{v}_ne^{\lambda_n t}\}$$

is a set of solutions. Moreover, the functions above are linearly independent since they are just the images of the invertible matrix C of the fundamental set of solutions to $\vec{x}' = D\vec{x}$, already a linearly independent set. So they comprise a fundamental set of solutions and the theorem is proved. \square

Example 28.7. Solve the system $y_1' = 4y_1 - 6y_2$, $y_2' = 3y_1 - 7y_2$.

Solution. This corresponds to the matrix system $\vec{y}' = A\vec{y}$, where $\vec{y} = \begin{bmatrix} 4 & -6 \\ 3 & -7 \end{bmatrix}$. In a previous example, we computed that the eigenvalues of A are

$$\lambda_1 = -5, \lambda_2 = 2,$$

with corresponding eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

respectively. So by our previous theorem, a fundamental set of solutions to the system is exactly

$$\left\{ \left[\begin{array}{c} (2/3) \\ 1 \end{array} \right] e^{-5t}, \left[\begin{array}{c} 3 \\ 1 \end{array} \right] e^{2t} \right\} = \left\{ \left[\begin{array}{c} (2/3)e^{-5t} \\ e^{-5t} \end{array} \right], \left[\begin{array}{c} 3e^{2t} \\ e^{2t} \end{array} \right] \right\}.$$

Thus all possible solutions to the ODE are linear combinations of the above, i.e. a general solution is

$$\vec{y} = C_1 \left[\begin{array}{c} 2/3 \\ 1 \end{array} \right] e^{-5t} + C_2 \left[\begin{array}{c} 3 \\ 1 \end{array} \right] e^{2t}.$$

□

29 Solving Systems with Complex Eigenvalues

Theorem 29.1. *Let A be an $n \times n$ matrix with real entries, and suppose $a + bi$ is a complex eigenvalue of A with corresponding non-zero eigenvector $\vec{v}_1 + \vec{v}_2 i$, where \vec{v}_1 and \vec{v}_2 are real vectors. Then the complex conjugate $a - bi$ is also an eigenvalue of A , and the complex conjugate $\vec{v}_1 - \vec{v}_2 i$ is an eigenvector corresponding to $a - bi$.*

Proof. Note that since $\vec{v}_1 + \vec{v}_2 i$ is an eigenvector corresponding to $a + bi$, we have

$$A(\vec{v}_1 + \vec{v}_2 i) = (a + bi)(\vec{v}_1 + \vec{v}_2 i) = (a\vec{v}_1 - b\vec{v}_2) + (b\vec{v}_1 + a\vec{v}_2)i.$$

Since A has real entries and \vec{v}_1, \vec{v}_2 are real vectors, it follows that

$$A\vec{v}_1 = a\vec{v}_1 - b\vec{v}_2 \text{ and } A\vec{v}_2 = b\vec{v}_1 + a\vec{v}_2.$$

In that case,

$$A(\vec{v}_1 - \vec{v}_2 i) = (a\vec{v}_1 - b\vec{v}_2) - (b\vec{v}_1 + a\vec{v}_2)i = (a - bi)(\vec{v}_1 - \vec{v}_2 i)$$

and therefore $\vec{v}_1 - \vec{v}_2 i$ is an eigenvector corresponding to $a - bi$ as claimed. □

Theorem 29.2. *Let A be an $n \times n$ matrix with real entries, and suppose $a + bi, a - bi$ are distinct complex conjugate eigenvalues of A with corresponding complex conjugate eigenvectors $\vec{v}_1 + \vec{v}_2 i, \vec{v}_1 - \vec{v}_2 i$ respectively. Then two linearly independent real vector-valued solutions of the ODE $\vec{y}' = A\vec{y}$ are given by*

$$e^{at}(\vec{v}_1 \cos(bt) - \vec{v}_2 \sin(bt))$$

and

$$e^{at}(\vec{v}_2 \cos(bt) + \vec{v}_1 \sin(bt)).$$

Proof. First note that the complex vector-valued function $\vec{y}_1 = (\vec{v}_1 + \vec{v}_2 i)e^{(a+bi)t}$ is a solution to the ODE: since \vec{v} is an eigenvector corresponding to $a + bi$, we have $A\vec{y} = A\vec{v}e^{(a+bi)t} = (a + bi)\vec{v}e^{a+bi} = \vec{y}'$. Now expand this solution \vec{y}_1 using Euler's formula:

$$\begin{aligned} \vec{y}_1 &= (\vec{v}_1 + \vec{v}_2 i)e^{(a+bi)t} \\ &= (\vec{v}_1 + \vec{v}_2 i)e^{at}(\cos bt + i \sin bt) \\ &= e^{at}(\vec{v}_1 \cos bt - \vec{v}_2 \sin bt) + ie^{at}(\vec{v}_2 \cos bt + \vec{v}_1 \sin bt). \end{aligned}$$

Since the ODE is linear, both the real and imaginary parts of the above have to be solutions to the ODE as well as \vec{y}_1 . (We will not repeat the usual argument here- see Theorem 14.5 to remember how it

goes.) This proves the theorem. (One may check that the same pair of solutions falls out when repeating the argument with $a - bi$ and $\vec{v}_1 - \vec{v}_2 i$ instead of $a + bi$ and $\vec{v}_1 + \vec{v}_2 i$, as we have done.) \square

Example 29.3. Find a general solution to $\vec{y}' = A\vec{y}$, where $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

Solution. First we compute eigenvalues and eigenvectors of A . The characteristic polynomial of A is

$$\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) - 4(-2) = \lambda^2 - 2\lambda + 5.$$

The quadratic formula yields $\lambda = 1 \pm 2i$ as roots to the above, so $1 + 2i$ and $1 - 2i$ are the complex conjugate eigenvalues of A .

To apply our previous theorem and solve the ODE, we actually only need to find an eigenvector corresponding to $1 + 2i$. So we consider the matrix

$$A - (1 + 2i)I = \begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix}.$$

The above row-reduces to $\begin{bmatrix} 8 & -4 + 4i \\ 4 & -2 - 2i \end{bmatrix}$, and then to $\begin{bmatrix} 2 & -1 + i \\ 0 & 0 \end{bmatrix}$. The null-space of this matrix is all vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ for which $2v_1 + (-1 - i)v_2 = 0$, i.e. for which $v_1 = (\frac{1}{2} + \frac{1}{2}i)v_2$. Thus an eigenvector corresponding to $1 + 2i$ is

$$\begin{bmatrix} 1/2 + (1/2)i \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} i.$$

Lastly, we apply our previous theorem to determine that a pair of linearly independent solutions to $\vec{y}' = A\vec{y}$ is given by

$$\left\{ e^t \left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \sin 2t \right), e^t \left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \sin 2t \right) \right\}.$$

Since A is 2×2 , the above is actually a complete basis for the solution space. So a general solution has the form

$$\vec{y} = C_1 e^t \left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \sin 2t \right) + C_2 e^t \left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \sin 2t \right).$$

\square

Example 29.4. Solve $\vec{y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \vec{y}$.

Solution. The characteristic polynomial of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ is $\det(A - \lambda I) = (1 - \lambda)[(1 - \lambda)^2 - 1(-1)] = -(\lambda^3 - 3\lambda^2 + 4\lambda - 2) = -(\lambda - 1)(\lambda - 2\lambda + 2)$. This polynomial has a real root $\lambda_1 = 1$, and two complex conjugate roots $\lambda_2 = 1 + i$, $\lambda_3 = 1 - i$. These are the eigenvalues of A .

For the real eigenvalue $\lambda_1 = i$, compute an eigenvector: we have $A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, which row-reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix will send a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ to $\vec{0}$ if and only if $v_2 = v_3 = 0$. So a good choice of non-zero eigenvector is, for instance, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This yields a solution $\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$ to the ODE.

For the complex conjugate roots $1 \pm i$, we compute an eigenvector corresponding to $1 + i$. We have $A - (1 + i)I = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{bmatrix}$, which row-reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$. For $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ to be in the null space of the latter matrix, we need $v_1 = 0$ and $v_2 - iv_3 = 0$, i.e. $v_2 = iv_3$. Choosing $v_3 = 1$, we get $\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} i$ as such a vector.

This yields the two functions $\vec{y}_2 = e^t \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \sin t \right)$ and $\vec{y}_3 = e^t \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t \right)$ as solutions to the ODE. The collection $\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ are linearly independent, so we have found a fundamental set of solutions. A general solution looks like:

$$\vec{y} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + C_2 e^t \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \sin t \right) + C_3 e^t \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t \right).$$

□

Technique 29.5 (General Strategy for Solving Linear Homogeneous Systems with Constant Coefficients). (1)

Given a system $\vec{y}' = A\vec{y}$, find the eigenvalues $\lambda_1, \dots, \lambda_n$ of A , which are precisely the roots of the characteristic polynomial $\det(A - \lambda I)$. Assume all of the eigenvalues are distinct.

- (2) For each real eigenvalue λ_i , find a corresponding non-zero eigenvector \vec{v}_i . This may be done by row reducing the matrix $A - \lambda_i I$ and picking a vector \vec{v}_i from its null space. Then a corresponding solution to the ODE is $\vec{y}_i = \vec{v}_i e^{\lambda_i t}$.
- (3) For each pair of complex conjugate eigenvalues $\lambda_i = a + bi$, $\lambda_{i+1} = a - bi$, find a non-zero eigenvector \vec{v}_i corresponding to $a + bi$. Again, this may be done by row reducing $A - \lambda_i I$ and picking a vector \vec{v}_i from its null space. This \vec{v}_i will be complex, so write $\vec{v}_i = \vec{v}_1 + \vec{v}_2 i$. Then two solutions to the ODE are $\vec{y}_i = e^{at}(\vec{v}_1 \cos bt - \vec{v}_2 \sin bt)$ and $\vec{y}_{i+1} = e^{at}(\vec{v}_2 \cos bt + \vec{v}_1 \sin bt)$.
- (4) The solutions $\{\vec{y}_1, \dots, \vec{y}_n\}$ obtained in parts (2) and (3) above comprise a linearly independent set, and thus are a fundamental set of solutions to the system. Therefore a general solution is $\vec{y} = C_1 \vec{y}_1 + \dots + C_n \vec{y}_n$.

Remark 29.6. Although the technique above goes a long way, we have not completely solved the problem of finding solutions to ODE's of the form $\vec{y}' = A\vec{y}$ where A is a real matrix, for the following reason: it is possible for A to have *repeated* eigenvalues λ_i , i.e. roots of the characteristic polynomial $\det(A - \lambda I)$ with multiplicity $k > 1$. When this happens, the situation is a bit complicated because it breaks into two cases. In one case, the repeated eigenvalue λ_i actually has k many linearly independent eigenvectors (matching its multiplicity of k). This is the easy case and we proceed to give solutions as in the technique above. In the other case, however, it is possible for an eigenvalue λ_i with multiplicity k to have strictly fewer than k linearly independent eigenvectors. In this case, a bit more work remains to be done to find k many solutions to the ODE. Unfortunately we have run out of time for the semester! Feel free to ask me about this if you are curious, and the details will probably be added to a future incarnation of the notes.