Final Exam Fall 2015

1) Evaluate $\int x^3\sqrt{1-x^2}\,dx$

<u>Solution:</u>

$$\left.\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\, d\theta \end{array}\right\} \Rightarrow \int \sin^3\theta \cos\theta \cdot \cos\theta\, d\theta$$

$$= \int \sin^3\theta\cos^2\theta\, d\theta = \int \sin\theta \sin^2\theta \cos^2\theta\, d\theta$$

$u=\cos\theta$

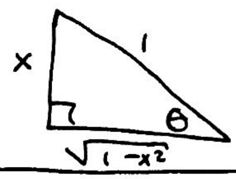
$du=-\sin\theta\, d\theta$

$$= \int (1-u^2)u^2\,du = \int u^2-u^4\,du = \frac{u^3}{3}-\frac{u^5}{5}+C$$

$$= \frac{\cos^3(\theta)}{3}-\frac{\cos^5(\theta)}{5}+C$$

$$= \frac{(\sqrt{1-x^2})^3}{3}-\frac{(\sqrt{1-x^2})^5}{5}+C$$

$\frac{x}{1}=\sin\theta$



2) Compute $\int_{-1}^{1}\frac{1}{x^5}\,dx$

<u>Solution:</u> We have an issue at $x=0$, so we take limits,

$$= \lim_{R\to 0^-}\int_{-1}^{R}\frac{1}{x^5} + \lim_{R\to 0^+}\int_{R}^{1}\frac{1}{x^5}\,dx$$

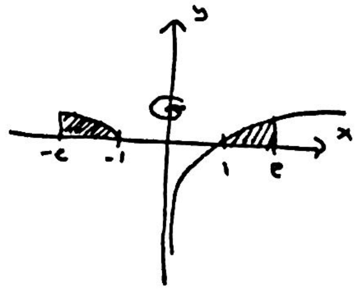
$$= \lim_{R\to 0^-}\left[\frac{1}{-4x^4}\right|_{-1}^{R} + \lim_{R\to 0^+}\left[\frac{1}{-4x^4}\right|_{R}^{1}$$

$$= \lim_{R\to 0^-}\left(-\frac{1}{4R^4}+\frac{1}{4}\right) + \lim_{R\to 0^+}\left(-\frac{1}{4}+\frac{1}{4R^4}\right)$$

$= -\infty \quad + \infty$, so integral diverges.

3) Let R be region enclosed by $y=2\ln x$, x-axis, and lines $x=1$ and $x=e$. Find volume of solid when revolved around y-axis.

Seems logical to use shell method. Know, in that case, that $V=2\pi\int_a^b(\text{radius})(\text{height})$



$$= 2\pi\int_1^e x(\ln x)\,dx$$

$u=\ln x \quad dv=x\,dx$

$du=\frac{1}{x}dx \quad v=\frac{x^2}{2}$

$$= 2\pi\left[\frac{\ln x(x^2)}{2}\right|_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx\right]$$

$$= 2\pi\left[\frac{e^2}{2}-0-\frac{1}{2}\int_1^e x\,dx\right] = 2\pi\left(\frac{e^2}{2}-\frac{1}{2}\left(\frac{x^2}{2}\right|_1^e\right)$$

$$= 2\pi\left(\frac{e^2}{2}-\frac{1}{4}\left(e^2-\frac{1}{2}\right)\right)$$

$$= 2\pi\left(\frac{e^2}{4}+\frac{1}{8}\right).$$

4) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}\right)$

First, apply alternating series test. If $a_n = \frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}$, we notice that

$\lim_{n\to\infty} a_n = 0$ and a_n is decreasing (pick your favorite method to show that),

so $\sum_{n=1}^{\infty}(-1)^n\left(\frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}\right)$ converges by A.S.T. Now, to test for absolute convergence,

Check $\sum_{n=1}^{\infty}\left|(-1)^n\left(\frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}\right)\right| = \sum_{n=1}^{\infty}\frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}$. Note, it acts like $\frac{1}{n}=b_n$,

So, for limit comparison, $\lim_{n\to\infty}\frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}\cdot\frac{n}{1} = 8/4 = 2$, hence, since $\sum b_n$ diverges, then $\sum_{n=1}^{\infty}\frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}$ diverges, so $\sum_{n=1}^{\infty}(-1)^n\left(\frac{8n^{3/2}-\sqrt{n}}{4n^{5/2}-5n+1}\right)$ converges conditionally.

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5) Determine I.O.C. of $\sum_{k=1}^{\infty}\frac{(x+1)^k}{3^k\cdot k^5}$

Ratio test! $\rho = \lim_{k\to\infty}\left|\frac{(x+1)^{k+1}}{3^{k+1}(k+1)^5}\cdot\frac{3^k k^5}{(x+1)^k}\right| = \lim_{k\to\infty}\left|\frac{(x+1)}{3}\cdot\left(\frac{k}{k+1}\right)^5\right| = \left|\frac{x+1}{3}\right| < 1$

$\Rightarrow |x+1| < 3$

$\Rightarrow -3 < x+1 < 3$

$\Rightarrow -4 < x < 2$

Check endpoints,

$\underline{-4}$: $\sum_{k=1}^{\infty}\frac{(-3)^k}{3^k k^5} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^5}$ which converges absolutely by p-series.

$\underline{2}$: $\sum_{k=1}^{\infty}\frac{3^k}{3^k k^5} = \sum_{k=1}^{\infty}\frac{1}{k^5}$ which conv. abs. by p-series,

hence interval of convergence is $[-4, 2]$.

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6) Find 5th degree term in Maclaurin exp. for x^2e^{2x}

Solution: We know (I know) $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$, $x\in\mathbb{R}$

Hence, $e^{2x} = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \cdots$, $2x\in\mathbb{R} \Rightarrow x\in\mathbb{R}$

so $x^2e^{2x} = \sum_{n=0}^{\infty}\frac{2^n x^{n+2}}{n!} = x^2 + 2x^3 + \frac{4x^4}{2!} + \frac{8x^5}{3!}$

$= x^2 + 2x^3 + 2x^4 + \frac{8x^5}{3!}$

7) Find Maclaurin rep for $\cos^2 x$.

Know $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, and $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$, $x \in \mathbb{R}$

$$\Rightarrow \cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$$

So $\cos^2 x = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$

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8) Compute $\sum_{k=2}^{\infty} \frac{5 \cdot (-1)^k}{6^k}$

This is a geo. series, so

$$\sum_{k=2}^{\infty} \frac{5 \cdot (-1)^k}{6^k} = 5 \sum_{k=2}^{\infty} \left(-\tfrac{1}{6}\right)^k = 5\left(\frac{(-1/6)^2}{1+1/6}\right) = 5\left(\frac{\frac{1}{36}}{\frac{7}{6}}\right) = 5\left(\frac{1}{42}\right) = 5/42$$