Final Exam Spring 2015

1) $$\sum_{n=1}^{\infty} (-1)^n \frac{12n^2-5n}{3n^4-5n+1}$$

Solution: Typically, one would first apply the Alternating Series Test, since we have the negative term, but with a small amount of foresight, we save time.

Consider

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{12n^2-5n}{3n^4-5n+1}\right| = \sum_{n=1}^{\infty}\frac{12n^2-5n}{3n^4-5n+1}.$$ If this series converges, our original series converges absolutely. Let's use the Limit Comparison Test. Let

$a_n = \frac{12n^2-5n}{3n^4-5n+1}$. Note, a_n acts a lot like $\frac{n^2}{n^4} = \frac{1}{n^2}$, so let's choose $b_n = \frac{1}{n^2}$. Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{12n^2-5n}{3n^4-5n+1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{12n^4-5n^3}{3n^4-5n+1} = 12/3 = 4$. Since the limit is positive and finite, and $\sum \frac{1}{n^2}$ converges by p-series, $\sum_{n=1}^{\infty} \frac{12n^2-5n}{3n^4-5n+1}$ converges by L.C.T., so our original series converges absolutely by definition.

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2) $$\int \frac{2x-1}{x^3-5x^2+6x}\,dx = \int \frac{2x-1}{x(x^2-5x+6)}\,dx = \int \frac{2x-1}{x(x-3)(x-2)}\,dx$$

P.F.: $$\frac{2x-1}{x(x-3)(x-2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-2}$$

$\Rightarrow$ $2x-1 = A(x-3)(x-2) + Bx(x-2) + Cx(x-3)$. All coefficients hold $\forall x$, so we choose "any" x,

$\underline{x=3}$

$5 = B(3)(3-2) = 3B$

$\boxed{B = 5/3}$

$\underline{x=0}$:

$-1 = A(-3)(-2)$

$-1 = 6A$

$\boxed{A = -1/6}$

$\underline{x=2}$:

$3 = C(2)(2-3)$

$3 = -2C$

$\boxed{C = -3/2}$

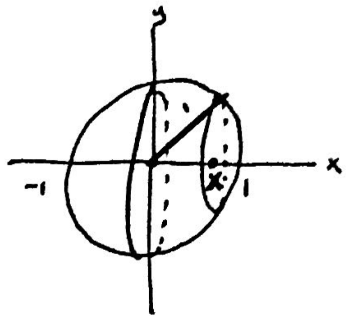
Hence, $$\int \frac{2x-1}{x(x-3)(x-2)}\,dx = \frac{-1}{6}\int \frac{1}{x}\,dx + \frac{5}{3}\int \frac{1}{x-3}\,dx - \frac{3}{2}\int \frac{1}{x-2}\,dx$$

$$= -\frac{1}{6}\ln|x| + \frac{5}{3}\ln|x-3| - \frac{3}{2}\ln|x-2| + C.$$ [NOTE: The integral is a u-sub, be careful when evaluating]

3) Volume of ☺ radius 1 (High school says we should get $\frac{4\pi}{3}$)

Solution: We use the cross sectional area format, hence

$$V = \int_a^b A(x)\,dx \quad \text{where} \quad A(x) \text{ is cross sectional area.}$$



A cross section of the sphere, seen in the picture, is a circle, namely one whose radius r satisfies

$x^2 + r^2 = 1$

$\Rightarrow \quad r^2 = 1 - x^2$

$\Rightarrow \quad r = \sqrt{1-x^2}.$

We know Area of a circle is $A = \pi r^2 = \pi(1-x^2)$, hence volume is

$$V = \int_{-1}^{1} \pi(1-x^2)\,dx$$

$$= \pi\left[x - \frac{x^3}{3}\right|_{-1}^{1} = \pi\left[1 - \tfrac{1}{3} - \left(-1 + \tfrac{1}{3}\right)\right] = \pi\left[2 - \tfrac{2}{3}\right] = \frac{4\pi}{3}.$$

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4) Compute $\sum_{n=1}^{\infty} \frac{10 \cdot (-2)^n}{3^n}$

Note, $\sum_{n=1}^{\infty} \frac{10 \cdot (-2)^n}{3^n} = 10 \cdot \sum_{n=1}^{\infty} \left(-\tfrac{2}{3}\right)^n$, and we know the formula for a geometric series,

so $10 \cdot \sum_{n=1}^{\infty} \left(-\tfrac{2}{3}\right)^n = 10\left(\frac{(-2/3)^1}{1 + 2/3}\right) = 10\left(\frac{-2/3}{5/3}\right) = 10\left(-\tfrac{2}{5}\right) = -4.$

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5) Evaluate $\int \frac{1}{x^2\sqrt{x^2-36}}\,dx$. #trig sub

Know $\tan^2 x + 1 = \sec^2 x$, so

let $x = 6\sec\theta$

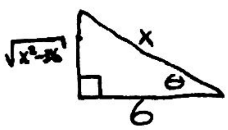
$dx = 6\sec\theta\tan\theta\,d\theta$

$$\Rightarrow \int \frac{6\sec\theta\tan\theta}{36\sec^2\theta \cdot 6\tan\theta}\,d\theta = \frac{1}{36}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{36}\int \cos\theta\,d\theta$$

$$= -\frac{1}{36}\sin\theta + C$$

Now, we need to go back to x's. To do this, only use the portion $x = 6\sec\theta$. No magic.

$\Rightarrow \sec\theta = \frac{x}{6} = \frac{\text{hyp}}{\text{adj}}$



Hence, $\int \frac{1}{x^2\sqrt{x^2-36}}\,dx = -\frac{1}{36}\sin\theta + C = \frac{-1}{36} \cdot \left(\frac{\sqrt{x^2-36}}{x}\right) + C.$

6) Compute $\int_0^\infty xe^{-x}\,dx$.

<u>Solution</u>: We have an improper integral, so we take limits,

$$\lim_{R\to\infty}\int_0^R xe^{-x}\,dx$$, then proceed with int. by parts,

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= \lim_{R\to\infty} -xe^{-x}\Big|_0^R + \lim_{R\to\infty}\int_0^R e^{-x}dx = \lim_{R\to\infty} -xe^{-x}\Big|_0^R + \lim_{R\to\infty} -e^{-x}\Big|_0^R$$

$$= \lim_{R\to\infty} -Re^{-R} + 0 + \lim_{R\to\infty} -e^{-R} + e^0$$

$$= \lim_{R\to\infty} \frac{-R}{e^R} + \lim_{R\to\infty} -\frac{1}{e^R} + 1$$

L'Hop

$$= \lim_{R\to\infty} \frac{-1}{e^R} + \lim_{R\to\infty} \frac{-1}{e^R} + 1 = 1.$$

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7) Determine I.O.C. for $\sum_{k=0}^\infty \frac{(x+10)^k}{k\cdot 4^k}$

First thought is (and likely should be) the ratio test.

$$\rho = \lim_{k\to\infty}\left|\frac{\frac{(x+10)^{k+1}}{(k+1)4^{k+1}}}{\frac{(x+10)^k}{k\cdot 4^k}}\right| = \lim_{k\to\infty}\left|\frac{(x+10)^{k+1}}{(k+1)4^{k+1}}\cdot\frac{k\cdot 4^k}{(x+10)^k}\right| = \lim_{k\to\infty}\left|\frac{(x+10)}{4}\cdot\left(\frac{k}{k+1}\right)\right|$$

$$= \left|\frac{x+10}{4}\right| < 1$$

$\Rightarrow |x+10| < 4$

$\Rightarrow -4 < x+10 < 4$

$\Rightarrow -14 < x < -6$

Need to check endpoints:

<u>-6</u>: $\sum_{k=0}^\infty \frac{4^k}{k\cdot 4^k} = \sum_{k=0}^\infty \frac{1}{k}$ which diverges (harmonic series).

<u>-14</u>. $\sum_{k=0}^\infty \frac{(-4)^k}{k\cdot 4^k} = \sum_{k=0}^\infty \frac{(-1)^k}{k}$ which we know converges conditionally.

Hence IOC is $[-14, -6)$.

8) Find 3^{rd} degree term in Taylor exp. for $\ln x$ centered at $c = 2$

Solution:

Know $T(x) = f(c) + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$

So, if

$$\left.\begin{aligned} f(x) &= \ln x \\ f'(x) &= \frac{1}{x} = x^{-1} \\ f''(x) &= -x^{-2} = -\frac{1}{x^2} \\ f'''(x) &= 2x^{-3} = \frac{2}{x^3} \end{aligned}\right\} \Rightarrow \begin{aligned} f(c) &= \ln(2) \\ f'(c) &= \frac{1}{2} \\ f''(c) &= -1/4 \\ f'''(c) &= \frac{2}{8} = 1/4 \end{aligned}$$

So $T_3(x) = \ln(2) + \frac{1}{2}(x-2) + \frac{-(x-2)^2}{4 \cdot 2!} + \frac{1}{4 \cdot 3!}(x-2)^3$

$\Rightarrow T_3(x) = \ln(2) + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{4!}$