MATH 166 TEST 3 REVIEW SHEET

General Comments and Advice: The student should regard this review sheet *only* as a sample of potential test problems, and not an end-all-be-all guide to its content. Anything and everything which we have discussed in class is fair game for the test. The test will cover roughly Sections 10.1, 10.2, 10.3, 10.4, and 10.5. Don't limit your studying to this sheet; if you feel you don't fully understand a particular topic, then try to do more problems out of your textbook!

Summary of Relative Growth Rates. For two sequences (a_n) and (b_n) such that $(a_n) \to \infty$ and $(b_n) \to \infty$, let the symbols $(a_n) << (b_n)$ denote that (b_n) grows faster than (a_n). Then the following is a summary of our most commonly encountered growth rates:

$$(\ln n) << ((\ln n)^r) << (n^q) << (n) << (n^p) << (b^n) << (n!) << (n^n),$$

where $r > 1$, $0 < q < 1$, $p > 1$, and $b > 1$.

General Strategy for Determining Convergence of a Series. There is no hard and fast rule for determining whether a series converges or diverges, and in general the student should build up their own intuition on the subject through practice. However, the following mental checklist can be a helpful guideline.

(1) **Divergence Test.** Given a series $\displaystyle\sum_{n=1}^{\infty} a_n$, the first thing one should always ask is, do the terms $(a_n) \to 0$? If not, we are done and the series diverges by the DT. If the terms do shrink to zero, however, we have to put more thought into the problem.

(2) **Very Nice Series.** Is the series a p-series with $p > 1$, or a geometric series, or a telescoping series? If so we are done and the series converges by our general theorems. (Note that sometimes a telescoping series can be "hidden": for example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is a telescoping series in disguise. The method of partial fractions may reveal these.)

(3) **Alternating Series Test.** If the series is alternating and the terms go to zero, then the series converges by the AST.

(4) **Integral Test.** Do the terms of the series look like a function we could probably integrate? (As an example, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(1+n^2)^2}$ shouts "Substitution Rule" when we consider its improper integral analogue $\int_1^{\infty} 2x(1+x^2)^{-2}dx$. Remember the series and the improper integral are not equal to one another, but they do either both converge or both diverge.)

(5) **Direct Comparison Test and Limit Comparison Test.** Do the terms a_n of the series behave similarly to those of one of the Very Nice Series? If

so, a comparison test may be appropriate. Usually the LCT is more powerful than the DCT. (For examples, the series $\sum_{n=1}^{\infty} \frac{n}{n^3+5n^2-17}$ seems to behave like $\sum_{n=1}^{\infty} \frac{1}{n^2}$ for very large n and so we think it probably converges; conversely the series $\sum_{n=1}^{\infty} \frac{3n^7+100}{n^8}$ seems to behave like a constant multiple of the harmonic series for very large n, so we think it diverges. Usually the natural choices for a comparison test are a p-series or a geometric series.)

(6) **Ratio Test.** Are we looking at a series of fractions where the question of convergence seems to come down to comparing the growth rate of the numerator vs. the denominator? e.g. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ or $\sum_{n=1}^{\infty} \frac{n}{(\ln n)^4}$ or $\sum_{n=1}^{\infty} \frac{1000^n}{n!}$. Then Ratio Test has a good chance of working. (Ratio Test is almost always the most natural choice if there is a factorial $n!$ term appearing anywhere in the series.)

(7) **Root Test.** Are we looking at a series where "to the n-th power" occurs in a significant way? e.g. $\sum_{n=0}^{\infty} \left(\frac{2}{5n+1}\right)^n$ or $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$. Root Test might be easiest.

(8) **Words of Warning.** Your first attempt at a test may not be the right choice. Be aware of when a test fails (for instance if the limit in Ratio/Root Test is exactly 1, or extremal cases of Limit Comparison Test where limit is 0 or ∞, etc.) and be prepared to try a different strategy. On that note, it is often very helpful to do some private scratchwork where you determine for yourself if a particular test solves the problem, and *then* carefully write a solution for full credit in a homework, quiz, or exam setting.

1. Determine if the sequence converges, and if so, state to what limit.

(a) $\left(\frac{2n-3}{4n}\right)_{n=1}^{\infty}$

(b) $(13\arctan(1000n))_{n=1}^{\infty}$

(c) $\left(\frac{e^k}{k^{1000}}\right)_{k=1}^{\infty}$

(d) $(\sqrt[3]{1+n^3})_{n=1}^{\infty}$

(e) $\left(\frac{\arctan n}{n}\right)_{n=1}^{\infty}$

(f) $\left(\frac{1}{\sqrt[7]{1-n^7}}\right)_{n=1}^{\infty}$

(g) $(a_k)_{k=1}^{\infty}$, where $a_1 = \frac{3}{10}$ and $a_k = a_{k-1} + \frac{3}{10^k}$ for every $k \geq 1$

(h) $(b_k)_{k=1}^{\infty}$ where b_k is the k-th digit after the decimal point in the decimal expansion of $\pi - 3$

(i) $((1 + \frac{k}{n})^n)$, where k is a fixed real constant.

2. Compute the following series, if they converge.

(a) $\displaystyle\sum_{k=0}^{\infty} e^{-k}$

(b) $\displaystyle\sum_{n=3}^{\infty} 5\left(\frac{1}{5}\right)^n$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+6)(k+7)}$

(d) $\frac{1}{3} + \frac{1}{5} + \frac{3}{25} + \frac{9}{125} + \frac{27}{625} + \ldots$

(e) $\displaystyle\sum_{n=1000}^{\infty} \frac{10n^3}{87n^3 + 100n}$

(f) $\displaystyle\sum_{k=2}^{\infty} \frac{5 \cdot (-1)^k}{6^k}$

(g) $\displaystyle\sum_{k=0}^{\infty} \frac{2 + 2^k}{5^k}$

3. State whether the following series converge or diverge. Carefully justify each answer.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{\sqrt{k}}{(\ln k)^{10}}$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{k^k}$

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-2n^2}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{4}{k^2 \ln k}$

(e) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+1}{2k}\right)^k$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n + \sqrt{n}}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{25(k+4)}$

(h) $\displaystyle\sum_{n=5}^{\infty} \frac{(n!)^2}{(2n)!}$

(i) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{3n^2}{100+5n^2}\right)^n$

(j) $\displaystyle\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^{k^2}$

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{n^8}{n^{11}+3}$

4. Determine whether the following series diverge, converge absolutely, or converge conditionally.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k^3}}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+10}$

(e) $\displaystyle\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{k!\cdot 2^k}{k^k}$

(g) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{15n^4+4n}{n^{14/3}+n^{11/3}}$

Answer Key
Disclaimer: This key was written quickly and may contain errors or typos! Please let me know if you have detected one.

1. a. $\frac{1}{2}$
b. $\frac{13\pi}{2}$
c. diverges
d. diverges
e. 0
f. 0
g. $\frac{1}{3}$
h. diverges since $\pi - 3$ is irrational!
i. e^k

2. a. $\frac{e}{e-1}$
b. $\frac{1}{20}$
c. $\frac{1}{7}$
d. $\frac{5}{6}$
e. diverges
f. $\frac{5}{42}$
g. $\frac{25}{6}$

3. a. diverges (Sugg: Divergence Test)
b. converges (Sugg: Ratio Test)
c. converges (Sugg: Integral Test)
d. converges (Sugg: Direct Comparison Test)
e. converges (Sugg: Root Test)
f. diverges (Sugg: Limit Comparison Test)
g. diverges (Sugg: Limit Comparison Test)
h. converges (Sugg: Ratio Test)
i. converges (Sugg: Root Test)
j. converges (Sugg: Root Test)
k. converges (Sugg: Direct Comparison Test)

4. a. converges conditionally (Sugg: AST, and p-series)
b. converges absolutely (p-series)
c. converges absolutely (Sugg: DCT)
d. diverges (Terms don't converge to 0!)
e. converges absolutely (geometric series)
f. converges absolutely (Sugg: Ratio Test)
g. converges conditionally (Sugg: AST, and LCT with p-series)