8 Rank of a matrix

We already know how to figure out that the collection $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$ is linearly dependent or not if each $\boldsymbol{v}_j \in \mathbf{R}^n$. Recall that we need to form the matrix $[\boldsymbol{v}_1 \mid \ldots \mid \boldsymbol{v}_k]$ with the given vectors as columns and see whether the row echelon form of this matrix has any free variables. If these vectors linearly independent (this is an abuse of language, the correct phrase would be "if the collection composed of these vectors is linearly independent"), then, due to the theorems we proved, their span is a subspace of $\mathbf{R}^n$ of dimension k, and this collection is a basis of this subspace. Now, what if they are linearly dependent? Still, their span will be a subspace of $\mathbf{R}^n$, but what is the dimension and what is a basis?

Naively, I can answer this question by looking at these vectors one by one. In particular, if $\boldsymbol{v}_1 \neq 0$ then I form $\mathcal{B} = (\boldsymbol{v}_1)$ (I know that one nonzero vector is linearly independent). Next, I add $\boldsymbol{v}_2$ to this collection. If the collection $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ is linearly dependent (which I know how to check), I drop $\boldsymbol{v}_2$ and take $\boldsymbol{v}_3$. If independent then I form $\mathcal{B} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ and add now third vector $\boldsymbol{v}_3$. This procedure will lead to the vectors that form a basis of the span of my original collection, and their number will be the dimension.

Can I do it in a different, more efficient way? The answer if "yes." As a side result we'll get one of the most important facts of the basic linear algebra.

First, consider the following matrix of dimension $k \times n$

$$\boldsymbol{B} = \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ b_{21} & \ldots & b_{2n} \\ \vdots & & \\ b_{k1} & \ldots & b_{kn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_k \end{bmatrix},$$

that is, I now use my vectors as *rows* and not *columns* of my matrix. I take a matrix $\boldsymbol{P}_{m \times k}$ and multiply $\boldsymbol{B}$ from the left $\boldsymbol{PB} = \boldsymbol{A}$. After looking at the product of two matrices I see that the first row of $\boldsymbol{A}$ is given as a linear combination of $\boldsymbol{v}_j$:

$$p_{11}\boldsymbol{v}_1 + p_{12}\boldsymbol{v}_2 + \ldots + p_{1k}\boldsymbol{v}_k,$$

note that all the vectors are *row* vectors in the expression above. The same true for any other row of $\boldsymbol{A}$.

Definition 8.1. *The row space of a matrix is the span of all the rows of this matrix.*

Note that, according to the introduced definition, the row space of $\boldsymbol{B}$ is exactly the subspace of $\mathbf{R}^n$, whose dimension and basis we need to find.

Using this definition I can conclude that the row space of $\boldsymbol{A}$ is a subspace of the row space of $\boldsymbol{B}$. Now assume that $\boldsymbol{P}$ is invertible, hence I can write that

$$\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A},$$

and hence in this case the row space of $\boldsymbol{A}$ is a subspace of the row space of $\boldsymbol{B}$. I conclude that

Proposition 8.2. *Multiplication by an invertible matrix from the left does not change the row space.*

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Recall that the process of putting a matrix into the reduced row echelon form is the same as multiplication by an invertible matrix. Hence, if $\boldsymbol{B}'$ is the reduced row echelon form of $\boldsymbol{B}$ and using the proposition above I conclude that $\boldsymbol{B}'$ and $\boldsymbol{B}$ have the same row space. Now I am ready to prove

Proposition 8.3. *For the reduced row echelon form of a matrix the nonzero rows are a basis of the row space.*

Proof. We actually already proved that the nonzero rows of $\boldsymbol{B}'$ span the row space since the zero rows do not contribute anything. All I need to show is that they are linearly independent. Instead of giving a full proof I will provide an example that should be generalized. Assume that I have only two pivots. If I drop all the other coordinates except these two, I will end with two vectors $(1,0)$ and $(0,1)$, since the entry of the second row must be zero under the pivot of the first row. They are clearly linearly independent, hence the whole rows are also linearly independent. ■

Definition 8.4. *The number of linearly independent rows of a matrix, i.e., the dimension of the row space, is called row rank of a matrix and denoted* $\operatorname{rank}\boldsymbol{A}$.

Example 8.5. Consider three vectors of $\mathbf{R}^4$:

$$\begin{aligned}\boldsymbol{v}_1 &= (1,2,2,1),\\ \boldsymbol{v}_2 &= (0,2,0,1),\\ \boldsymbol{v}_3 &= (-2,0,-4,3).\end{aligned}$$

What is a basis of $\operatorname{span}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$?

I form the following matrix and row reduce it

$$\begin{bmatrix}1 & 2 & 2 & 1\\ 0 & 2 & 0 & 1\\ -2 & 0 & -4 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Clearly I have three pivots, no zero rows, and hence 1) vectors $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$ are linearly independent, the row rank of the corresponding matrix is 3, I can take as a basis of their span either the original vectors, or, somewhat more convenient for further calculations, vectors

$$\begin{aligned}\boldsymbol{u}_1 &= (1,0,2,0),\\ \boldsymbol{u}_2 &= (0,1,0,0),\\ \boldsymbol{u}_3 &= (0,0,0,1).\end{aligned}$$

Which vectors are in the span? I have for an arbitrary $\boldsymbol{x} = (x_1,x_2,x_3,x_4)$:

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \alpha_1\boldsymbol{u}_1 + \alpha_2\boldsymbol{u}_2 + \alpha_3\boldsymbol{u}_3 = \begin{bmatrix}\alpha_1\\ \alpha_2\\ 2\alpha_1\\ \alpha_3\end{bmatrix},$$

and I conclude that for a vector to be in my span, it must satisfy the condition $x_3 = 2x_1$. For example vector $(1,0,2,5)$ is in my span, but $(1,0,3,5)$ is not.

Now, let me go back to the beginning of this lecture. Now I have two tests that allow me to conclude something about linear dependence or independence of given vectors. The first one deals with a matrix composed of my vectors as columns, the second one — with the matrix composed of my vectors as rows. Is there a deeper connection between the two?

For a given matrix $\boldsymbol{A}$, I, of course, can consider not only the row space (the span of rows), but also the column space (the span of columns). I will denote them $\operatorname{row}\boldsymbol{A}$ and $\operatorname{col}\boldsymbol{A}$ respectively. Let me define the *nullity* $\operatorname{nul}\boldsymbol{A}$ of a matrix $\boldsymbol{A}$ as the number of free variables in the row echelon form. Since I defined the row rank of the matrix as the dimension of the row space, i.e., the number of pivots, I immediately conclude that

$$\dim\operatorname{row}\boldsymbol{A}+\operatorname{nul}\boldsymbol{A}=\operatorname{rank}\boldsymbol{A}+\operatorname{nul}\boldsymbol{A}=n.$$

Note also that $\operatorname{row}\boldsymbol{A}=\operatorname{col}\boldsymbol{A}^\top$ and vice versa.

Theorem 8.6. *The row rank of $\boldsymbol{A}_{m\times n}$ is equal to the row rank of $(\boldsymbol{A}^\top)_{n\times m}$.*

Remark 8.7. The statement of the theorem can be rephrased as "the dimension of the row space coincide with the dimension of the column space." This is an absolutely counterintuitive fact, since in general the column and row spaces are subspaces of different vectors spaces! This also allows us to talk about just "rank" of a matrix, and not about "row rank."

Proof. I will consider $S=(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_m)$ and $\boldsymbol{A}_{m\times n}$, whose rows are exactly vectors $\boldsymbol{v}_j$. This implies that the columns of $(\boldsymbol{A}^\top)_{n\times m}$ are the same vectors $\boldsymbol{v}_j$. First I will show that $\operatorname{rank}\boldsymbol{A}=m$ if and only if $\operatorname{rank}\boldsymbol{A}^\top=m$. Indeed, $\operatorname{rank}\boldsymbol{A}=m$ if and only if S is linearly independent (by the definition of rank). Further, S is linearly independent if and only if, by the first test of independence, $\boldsymbol{A}^\top\boldsymbol{x}=0$ has only trivial solution, that is, $\operatorname{nul}\boldsymbol{A}^\top=0$. By the formula above, I have that $\operatorname{rank}\boldsymbol{A}^\top+\operatorname{nul}\boldsymbol{A}^\top=m$ and hence the assumption that $\operatorname{nul}\boldsymbol{A}^\top=0$ is equivalent that $\operatorname{rank}\boldsymbol{A}^\top=m$. This is exactly what I needed to show.

Now, assume that $\boldsymbol{A}$ has rank $k<m$. This literally means that there are k rows of $\boldsymbol{A}$ which are linearly independent, and I form matrix $\boldsymbol{B}_{k\times n}$ out of these rows. Using what I proved above, $\operatorname{rank}\boldsymbol{B}^\top=k$, but since $\boldsymbol{B}^\top$ consists of some parts of the rows of $\boldsymbol{A}^\top$ I have (think the last inequality out!)

$$\operatorname{rank}\boldsymbol{A}=k=\operatorname{rank}\boldsymbol{B}=\operatorname{rank}\boldsymbol{B}^\top\leq\operatorname{rank}\boldsymbol{A}^\top.$$

Therefore, for any matrix $\operatorname{rank}\boldsymbol{A}\leq\operatorname{rank}\boldsymbol{A}^\top$, using the same inequality for $\boldsymbol{A}^\top$ I finally conclude that

$$\operatorname{rank}\boldsymbol{A}=\operatorname{rank}\boldsymbol{A}^\top.$$

■

There is actually yet another equivalent definition of the rank. Specifically, if I am given a matrix $\boldsymbol{A}_{m\times n}$, I can always form square matrices of size $k\times k, k\leq m,n$ keeping exactly k rows and k columns of the original matrix. I will call the determinants of such matrices the *minors of order k*.

Exercise 1. Calculate the minors of the matrix

$$\begin{bmatrix}1&-1&0\\2&2&-4\end{bmatrix}.$$

Definition 8.8. *The determinant rank of a matrix is the biggest number k, such that there exists a non-zero minor of order k.*

Theorem 8.9. *The determinant rank coincides with the row rank and hence with the column rank.*

Proof. First, consider the case $k > \operatorname{rank} \boldsymbol{A}$. It follows that any k columns of $\boldsymbol{A}$ are linearly dependent, and hence any $k \times k$ matrix must have determinant zero.

Now we need to find at least one minor of order exactly equal to $\operatorname{rank} \boldsymbol{A}$. Let $\boldsymbol{k}$ be the number of pivots in the row reduced echelon form of $\boldsymbol{A}$. I hence have $\operatorname{rank} \boldsymbol{A} = k$. Choose in the original matrix $\boldsymbol{A}$ k rows and k columns that correspond exactly to the pivots in the row reduced echelon form of $\boldsymbol{A}$. By construction the determinant of this matrix is nonzero. ∎

Finally, using the notion of the rank one can formulate the following result, which we already discussed at length in the first part of our course. Its proof is left as an exercise.

Theorem 8.10. *The system of linear algebraic equations $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if* $\operatorname{rank} \boldsymbol{A} = \operatorname{rank}[\boldsymbol{A} \mid \boldsymbol{b}]$.