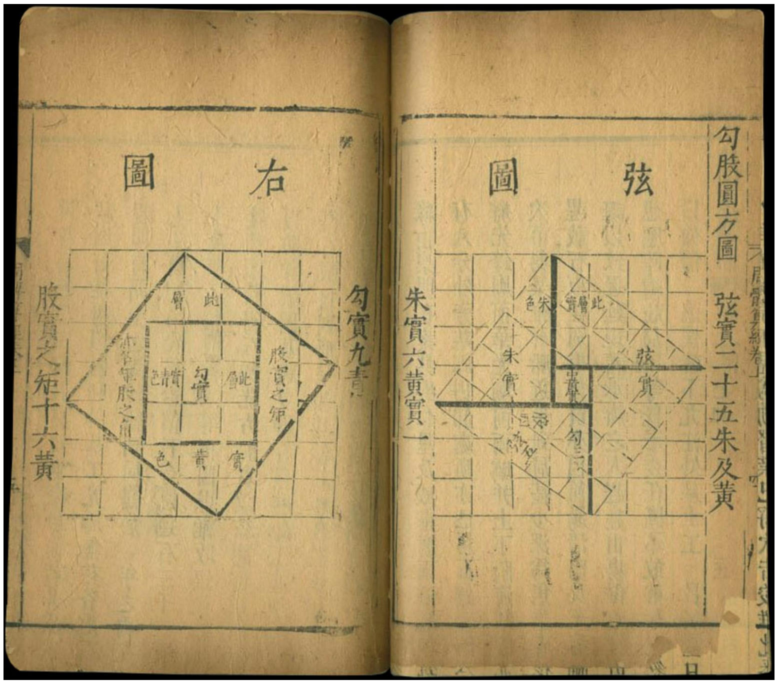
# 1 Topic: Pythagorean Theorem




The goal of this project is to learn two different proofs of Pythagorean Theorem. The first one is arguably most basic, requires only knowledge of properties of similar triangles and, most importantly, does not rely on the notion of "area" in any form. The second one I borrow from highly recommended book *Mathematics and Plausible Reasoning* by G. Polya. This proof necessarily will be more subtle since it is based on the notion of the area of a figure, but definitely is more conceptual and (from my point of view) much more elegant.

If you'd like to learn more about history of Pythagorean theorem (and about many other facts in the history of mathematics) with some interesting visual illustrations, please check the web page `https://www.history-of-mathematics.org/` that contains a ton of virtual exhibits.

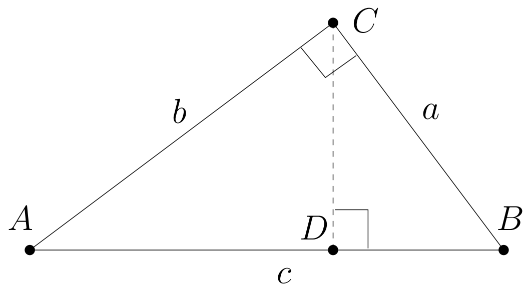
For the following you will need to recall the notion of *similarity.* Not very precisely: Two figures are *similar* if they have the same shape but different size. More precisely, two figures are *similar* if there is a one-to-one correspondence such that the distance between any pairs of corresponding points scales with the same factor (make a figure!), which is called the *coefficient of similarity.* For similar figures the corresponding angles are equal, straight lines correspond to straight lines, and the lengths of the corresponding sides have the matching ratio (which is exactly equal to the coefficient of similarity).

**Theorem 1.1** (Pythagorean Theorem)**.** *If in a right triangle the lengths of two legs are $a$ and $b$ and the length of hypotenuse is $c$ then*

$$a^2 + b^2 = c^2.$$

$\diamondsuit$ **1.1** (Proof 1)**.** This is Problem 3.3.15 from the textbook.

Consider the following sketch:




1. Prove that triangles $ACD$ and $CBD$ are both similar to triangle $ABC$.

2. For triangle $ABC$ with legs of lengths $a$ and $b$ and with hypotenuse of length $c$, use the proportionality of corresponding sides of similar triangles to establish that $a^2 + b^2 = c^2$.

3. Write down carefully your proof using the structure from lectures as an example.

♢ **1.2** (Proof 2)**.** To prepare for this proof, first you need to figure out how the ratio of the areas of two similar figures can be computed if the coefficient of similarity is $\lambda$ .

1. Let $ABC$ and $A_1B_1C_1$ be two similar triangles with the coefficient of similarity $\lambda$. Figure out how their areas are related and provide a proof of this fact.

2. Using the previous point make an educated guess what the ratio of areas of two similar figures is.

   Now we are ready to prove the theorem.

3. (Generalizing the theorem statement.) Prove that the sum of the areas of squares on the sides $a$, $b$ is equal to the area of the square on the side $c$ if and only if the sum of the areas of *any* two similar figures built on sides $a$ and $b$ is equal to the area of the third similar figure built on side $c$. (This point implies that we should not stick to squares and gives a much more general statement, it could be beneficial to try to make a sketch illustrating this point).

4. (Specializing) Identify three similar figures built on the sides $a$, $b$, and $c$, for which the theorem is "obvious," and hence conclude. *Hint:* Remember that the figures can be built not only "outside of the triangle" but also "inside it," and look at Proof 1 for inspiration.

5. Write down carefully your proof using the structure from lectures as an example.

♢ **1.3.** What is the converse statement to Pythagoras' theorem? If you think it is true provide a proof. If you think it is false give a counterexample. (Do not search online, do this problem yourself, you can use any general facts from your other math classes you want if necessary.)

♢ **1.4.** (Somewhat an open ended question.) Why in your opinion the Pythagorean Theorem is so important for the history of mathematics and, in particular, for the modern mathematics? (Recall, e.g., your Linear Algebra class and, if you took it, Math 346, Metric space topology.)