18 Green's function for the Poisson equation

Now that we have some experience working with Green's functions in one dimension, we are ready to see how Green's functions can be obtained in more physically realistic dimensions two and three. That is, in general I am looking to solve

$$-\Delta u = f, \quad \boldsymbol{x} \in D \subseteq \mathbf{R}^m, \quad m = 2, 3, \tag{18.1}$$

with the boundary conditions

$$u|_{\boldsymbol{x}\in\partial D} = 0, \tag{18.2}$$

(and the boundary condition can certainly be changed).

Using my intuition and experience from one dimensional case, I expect that to solve problem (18.1), (18.2) I need to find Green's function $G(\boldsymbol{x};\boldsymbol{\xi})$, i.e., the solution to

$$\begin{aligned} -\Delta G(\boldsymbol{x};\boldsymbol{\xi}) &= \delta(\boldsymbol{x}-\boldsymbol{\xi}), \quad \boldsymbol{x},\boldsymbol{\xi} \in D \subseteq \mathbf{R}^m, \quad m = 2, 3, \\ G(\boldsymbol{x};\boldsymbol{\xi}) &= 0, \quad \boldsymbol{x} \in \partial D. \end{aligned} \tag{18.3}$$

But before attacking problem (18.3), I will look into the problem *without* boundary conditions.

18.1 Fundamental solution to the Laplace equation

Definition 18.1. *Solution G_0 to the problem*

$$-\Delta G_0(\boldsymbol{x};\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}), \quad \boldsymbol{x},\boldsymbol{\xi} \in \mathbf{R}^m \tag{18.4}$$

is called the fundamental solution to the Laplace equation (or free space Green's function).

Planar case $m = 2$

To find G_0 I will appeal to the physical interpretation of my equation. Physically to solve (18.4) means to find *a potential* of the gravitational (or electrostatic) field, caused by the unit mass (unit charge) positioned at $\boldsymbol{\xi}$. The field itself is found as the gradient of G_0. Since I do not expect to have for my gravitation field any preferred directions, I conclude that my potential should only depend on the distance $r = |\boldsymbol{x}-\boldsymbol{\xi}|$ between the points $\boldsymbol{x}$ and $\boldsymbol{\xi}$ and not on any angle. Next, I will use the fact that G_0 satisfies the Laplace equation $\Delta G = 0$ at any point except $\boldsymbol{\xi}$. Using the polar form of the Laplace operator and the fact that my potential depends only on r, I get

$$rG_0'' + G_0' = 0.$$

I solved this equation before when I used the separation of variables for the Laplace equation in polar coordinates. The general solution is given by

$$G_0(r) = A \log r + B.$$

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Now I note that constant B will not contribute to the delta function, since it is infinitely differentiable, and hence my fundamental solution has the form $A\log r$, I need only to determine constant A. For this I will use the characteristic property of the delta function that

$$\int_{\mathbf{R}^2} \delta(\boldsymbol{x}-\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = 1,$$

and *the divergence theorem* that says that for a sufficiently nice domain D and smooth vector field $\boldsymbol{F}$

$$\int_D \nabla\cdot\boldsymbol{F}\,\mathrm{d}\boldsymbol{x} = \oint_{\partial D} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S,$$

where $\hat{\boldsymbol{n}}$, as before, is the outward normal to D.

Consider a disk D_ϵ or radius ϵ around $\boldsymbol{\xi}$. Then I have

$$\begin{aligned} 1 &= \int_{\mathbf{R}^2} \delta(\boldsymbol{x}-\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x} = \int_{D_\epsilon} \delta(\boldsymbol{x}-\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x} = \\ \text{[due to (18.4)]} &= -A\int_{D_\epsilon} \Delta\log r\,\mathrm{d}\boldsymbol{x} = -A\int_{D_\epsilon} \nabla\cdot\nabla\log r\,\mathrm{d}\boldsymbol{x} = \\ \text{[due to the divergence theorem]} &= -A\oint_{\partial D_\epsilon} \nabla\log r\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S = \\ \text{[why?]} &= -A\oint_{\partial D_\epsilon} \frac{\mathrm{d}\log r}{\mathrm{d}r}\,\mathrm{d}S = -A\int_0^{2\pi}\frac{1}{r}r\,\mathrm{d}\varphi = -A2\pi, \end{aligned}$$

whence

$$A = -\frac{1}{2\pi},$$

and therefore

$$G_0(\boldsymbol{x};\boldsymbol{\xi}) = -\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{\xi}| = -\frac{1}{4\pi}\log\big((x-\xi)^2+(y-\eta)^2\big)$$

is the fundamental solution to the planar Laplace equation or, physically, is the potential of the gravitation (or electrostatic) field induced by the unit mass (charge). Note that for the field itself

$$\frac{\mathrm{d}G_0}{\mathrm{d}r} = \frac{1}{r},$$

that is the force is inversely proportional to the distance between the points.

Case $m = 3$

Very briefly, and invoking exactly the same reasonings, I find that my fundamental solution must depend only on $r = |\boldsymbol{x}-\boldsymbol{\xi}|$ and solve everywhere except point $\boldsymbol{\xi}$ the equation

$$rG_0'' + 2G_0' = 0$$

(see one of the exercises in the very first lecture). The general solution to this equation is

$$\frac{A}{r} + B,$$

and therefore it is reasonable to assume that

$$G_0(r) = \frac{A}{r}\,.$$

Again, using the properties of delta function and the divergence theorem I get for the ball D_ϵ with the center at $\boldsymbol{\xi}$

$$\begin{aligned}
1 &= \int_{\mathbf{R}^3} \delta(\boldsymbol{x}-\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x} = \int_{D_\epsilon} \delta(\boldsymbol{x}-\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x} = \\
&= -A\int_{D_\epsilon} \Delta\frac{1}{r}\,\mathrm{d}\boldsymbol{x} = -A\int_{D_\epsilon} \nabla\cdot\nabla\frac{1}{r}\,\mathrm{d}\boldsymbol{x} = \\
&= -A\oint_{\partial D_\epsilon} \nabla\frac{1}{r}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S = \\
&= A\oint_{\partial D_\epsilon}\frac{1}{r^2}\,\mathrm{d}S = \frac{A}{\epsilon^2}\oint_{\partial D_\epsilon}\mathrm{d}S = \frac{A}{\epsilon^2}4\pi\epsilon^2 = 4\pi A,
\end{aligned}$$

since the area of the sphere of radius ϵ is $4\pi\epsilon^2$. Therefore, my fundamental solution is

$$G_0(r) = \frac{1}{4\pi r}\,,$$

and the gravitational (or electrostatic) field exerts the force that is inversely proportional to the square of the distance, as we all remember from our physics classes.

Exercise 1. Let f be a function on $\mathbf{R}^m$ of the distance $r = \sqrt{x_1^2 + \ldots + x_m^2}$ from the origin only. Show that $\nabla f\cdot\hat{\boldsymbol{n}}$ on the surface of the m-dimensional ball of radius ϵ can be found as

$$\nabla f\cdot\hat{\boldsymbol{n}} = \frac{\mathrm{d}f}{\mathrm{d}r}(r)|_{r=\epsilon} = f'(\epsilon).$$

Exercise 2. Find the fundamental solution to the Laplace equation for any dimension m.

18.2 Method of images to find Green's function

18.2.1 Green's function for the half-space

Now I can use my fundamental solutions to figure out Green's functions for certain domain with some symmetries. The key idea is very similar to the idea how I solved the wave equation on the half-line (recall that I used a *reflection* of the initial conditions in a way to satisfy the boundary condition automatically). Let me consider the problem in the domain $D = \{\boldsymbol{x} = (x, y) \in \mathbf{R}^2 \colon y > 0\}$, with the boundary condition given along the line $y = 0$:

$$\begin{aligned}
-\Delta u &= f, \quad \boldsymbol{x} \in D = \{\boldsymbol{x} = (x,y) \in \mathbf{R}^2 \colon y > 0\},\\
u &= 0, \quad \boldsymbol{x} \in \partial D.
\end{aligned} \tag{18.5}$$

My goal is to find Green's function, that solves the problem $-\Delta G = \delta(\boldsymbol{x}-\boldsymbol{\xi})$ in D for all $\boldsymbol{x}\boldsymbol{\xi} \in D$ (this is important!), and satisfies the given boundary condition $G = 0$ on ∂D. To find such function let me consider the problem

$$-\Delta G(\boldsymbol{x};\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}) - \delta(\boldsymbol{x}-\boldsymbol{\xi}^*),$$

where $\boldsymbol{\xi}^*$ is a point that *does not* belong to D. The ultimate goal is to find (guess) how the point $\boldsymbol{\xi}^*$ depends on the coordinates of the point $\boldsymbol{\xi}$ such that the boundary condition $G = 0$ on ∂D would be automatically satisfied. To get an inspiration, consider Fig. 1. Since in my problem the delta functions have opposite signs, it means that these two sources (of the heat, gravitational field, etc) will equilibrate each other at each point on the axis x if the distances to them are equal, but this is certainly true if for $\boldsymbol{\xi}^*$ I will take the point with the coordinates $(\xi, -\eta)$, i.e., simple reflection of the point $\boldsymbol{\xi}$ with respect to the x-axis.

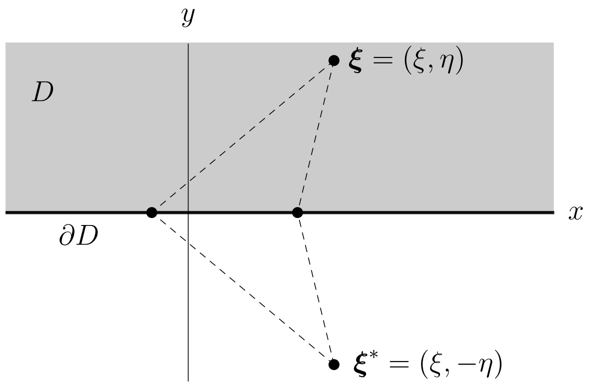

Figure 1: The construction of the image of the source with coordinates $\boldsymbol{\xi}$ for the half-plane.

Using the fundamental solution on the plane I have that my problem has the solution

$$\begin{aligned} G(\boldsymbol{x};\boldsymbol{\xi}) &= -\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{\xi}| + \frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{\xi}^*| \\ &= \frac{1}{2\pi}\log\left|\frac{\boldsymbol{x}-\boldsymbol{\xi}^*}{\boldsymbol{x}-\boldsymbol{\xi}}\right| \\ &= \frac{1}{4\pi}\log\frac{(x-\xi)^2+(y+\eta)^2}{(x-\xi)^2+(y-\eta)^2}, \end{aligned}$$

which is Green's function I am looking for. Indeed, in the domain D (again, abusing the language) $\delta(\boldsymbol{x}-\boldsymbol{\xi}^*)$ is zero, and hence in D function G solves $-\Delta G = \delta(\boldsymbol{x}-\boldsymbol{\xi})$, moreover, when $y = 0$ (plug it in) I get $G(\boldsymbol{x};\boldsymbol{\xi}) = 0$ for any point $\boldsymbol{x} = (x, 0)$, as required. Therefore, problem (18.5) is solved and its solution is given by

$$u(\boldsymbol{x}) = \int_D f(\boldsymbol{\xi})G(\boldsymbol{x};\xi)\,\mathrm{d}\boldsymbol{\xi}.$$

18.2.2 Green's function for the disk

In this section I will solve the Poisson equation in a disk of radius a. That is, I consider the problem

$$-\Delta u = f, \quad \boldsymbol{x} \in D \subseteq \mathbf{R}^2, \quad D = \{(x, y)\colon x^2 + y^2 < a^2\} \tag{18.6}$$

with the homogeneous Dirichlet or Type I boundary conditions

$$u|_{\boldsymbol{x}\in\partial D} = 0. \tag{18.7}$$

I know that to be able to write the solution to my problem, I need Green's function that solves

$$\begin{aligned} -\Delta G(\boldsymbol{x};\boldsymbol{\xi}) &= \delta(\boldsymbol{x}-\boldsymbol{\xi}), \quad \boldsymbol{x},\boldsymbol{\xi}\in D\subseteq \mathbf{R}^2,\\ G(\boldsymbol{x};\boldsymbol{\xi}) &= 0, \quad \boldsymbol{x}\in\partial D. \end{aligned} \tag{18.8}$$

If I am able to figure out the solution to (18.8), then (18.6), (18.7), by the principle of superposition, has the solution

$$u(\boldsymbol{x}) = \int_D f(\boldsymbol{\xi}) G(\boldsymbol{x};\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}.$$

The key idea is to replace problem (18.8) with another problem on the whole plane $\mathbf{R}^2$, with an additional source (or sources) outside of D, such that the boundary condition (18.7) would be satisfied automatically.

I again replace my problem (18.8) with the following

$$-\Delta G(\boldsymbol{x};\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}) - \delta(\boldsymbol{x}-\boldsymbol{\xi}^*), \quad \boldsymbol{x}\in\mathbf{R}^2,\ |\boldsymbol{\xi}|<a,\ |\boldsymbol{\xi}^*|>a. \tag{18.9}$$

Since I require the coordinates of my second source be outside of the my disk, hence *within* the disk, due to the properties of the delta function, (18.9) coincides with the equation (18.8). If I am capable to determine the coordinates of my second source as a function of the coordinates of the source inside the disk, such that for $|\boldsymbol{x}|=a$ my solution vanishes, then it means that I solved my problem. In other words, I am looking for the coordinates $\boldsymbol{\xi}^*$ of the *image* of the point $\boldsymbol{\xi}$, and this explains the name of the method.

So let me try to achieve my goal. I know that solution, again by the superposition principle, to (18.9) is given by

$$G(\boldsymbol{x};\boldsymbol{\xi}) = -\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{\xi}| + \frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{\xi}^*| + c = \frac{1}{4\pi}\log\frac{|\boldsymbol{x}-\boldsymbol{\xi}^*|^2}{|\boldsymbol{x}-\boldsymbol{\xi}|^2} + c.$$

I added an additional constant c here because it does not change anything (it is still the solution I am looking for), and yet it adds an additional degree of freedom in my calculations, without which I would not be able to achieve the required result.

Hence, for $|\boldsymbol{x}|=a$, I must have, due to (18.8),

$$|\boldsymbol{x}-\boldsymbol{\xi}|^2 = k|\boldsymbol{x}-\boldsymbol{\xi}^*|^2, \quad k = e^{4\pi c}.$$

To see whether the last equality must be true, I consider

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{\xi}|^2 &= (\boldsymbol{x}-\boldsymbol{\xi})\cdot(\boldsymbol{x}-\boldsymbol{\xi}) = |\boldsymbol{x}|^2+|\boldsymbol{\xi}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{\xi} = a^2 + r_0^2 - 2ar_0\cos\theta,\\ |\boldsymbol{x}-\boldsymbol{\xi}^*|^2 &= (\boldsymbol{x}-\boldsymbol{\xi}^*)\cdot(\boldsymbol{x}-\boldsymbol{\xi}^*) = |\boldsymbol{x}|^2+|\boldsymbol{\xi}^*|^2 - 2\boldsymbol{x}\cdot\boldsymbol{\xi}^* = a^2 + \gamma^2 r_0^2 - 2\gamma a r_0\cos\theta, \end{aligned}$$

where I assumed, to reduce the number of free parameters, that the angle θ between $\boldsymbol{x}$ and $\boldsymbol{\xi}$ and $\boldsymbol{x}$ and $\boldsymbol{\xi}^*$ is the same, that is $\boldsymbol{\xi}^*=\gamma\boldsymbol{\xi}$.

To get the required equality I must have

$$\begin{aligned} a^2 + r_0^2 &= ka^2 + k\gamma^2 r_0^2,\\ ar_0 &= k\gamma a r_0, \end{aligned}$$

from the second of which $k\gamma = 1$ and hence from the first

$$\gamma = \frac{a^2}{r_0^2}\,.$$

Problem solved! You can see geometrically that my point $\boldsymbol{\xi}^*$ is one of the vertices of the triangle $0\boldsymbol{x}\boldsymbol{\xi}^*$, which is similar by construction to the triangle $0\boldsymbol{x}\boldsymbol{\xi}$, see Fig. 2.

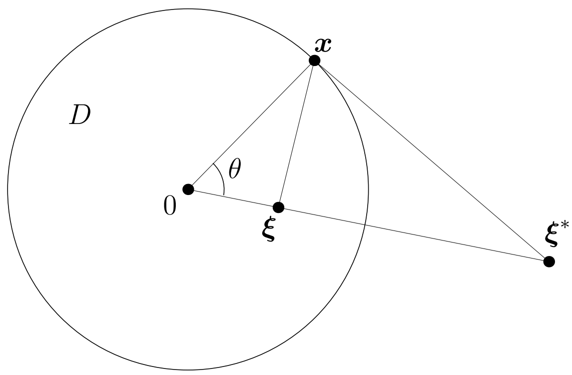

Figure 2: The construction of the image of the source with coordinates $\boldsymbol{\xi}$ for the disk.

Explicitly, I have

$$G(\boldsymbol{x};\boldsymbol{\xi}) = -\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{\xi}| + \frac{1}{2\pi}\log\left(\frac{|\boldsymbol{\xi}|}{a}\left|\boldsymbol{x}-\frac{a^2}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}\right|\right).$$

Note that this expression holds only for $\boldsymbol{\xi}\neq 0$. Taking a limit $|\boldsymbol{\xi}|\to 0$ I find (fill in the details) that

$$G(\boldsymbol{x};\boldsymbol{0}) = -\frac{1}{2\pi}\log|x| + \frac{1}{2\pi}\log a,$$

which is clearly satisfies the boundary condition.

Using the polar coordinates for the point $\boldsymbol{x}$ as (r,ϕ) and (r_0,ϕ_0) for $\boldsymbol{\xi}$, I can write my solution to (18.7), (18.8) in the polar form (check it)

$$G(r,\phi;r_0,\phi_0) = -\frac{1}{4\pi}\log\frac{a^2\big(r^2+r_0^2-2rr_0\cos(\phi-\phi_0)\big)}{r_0^2r^2+a^4-2a^2rr_0\cos(\phi-\phi_0)}\,.$$

Exercise 3. Find Green's function for the unit sphere.

Similar approach works for some other domains (see the homework problems), but the list of such domains is quite limited. There are other methods to infer the Green function, but they are outside of the scope of this introductory course. Probably still the best reference for a prepared reader to read about various methods to find Green's functions is the first volume of Courant and Hilbert *Methods of mathematical physics.*

18.3 Dirichlet's problem for the Laplace equation and Green's functions

18.4 Test yourself

18.5 Solutions to the exercises