14 Periodic phenomena in nature and limit cycles

14.1 Periodic phenomena in nature

As it was discussed while I talked about the Lotka–Volterra model, a great deal of natural phenomena show periodic behavior. Probably one of the best studied and advertised examples is the data from Hudson Bay Company that recorded the numbers of lynx and hare pelts that were bought by the company from hunters in the nineteenth and twentieth century. A canonical in some sense representation of these data is given in Figure 1, where the numbers of acquired skins of lynx (circles) and snowshoe hare (squares) are shown. The data show indisputable 10 year cycle in both the prey and predator numbers.

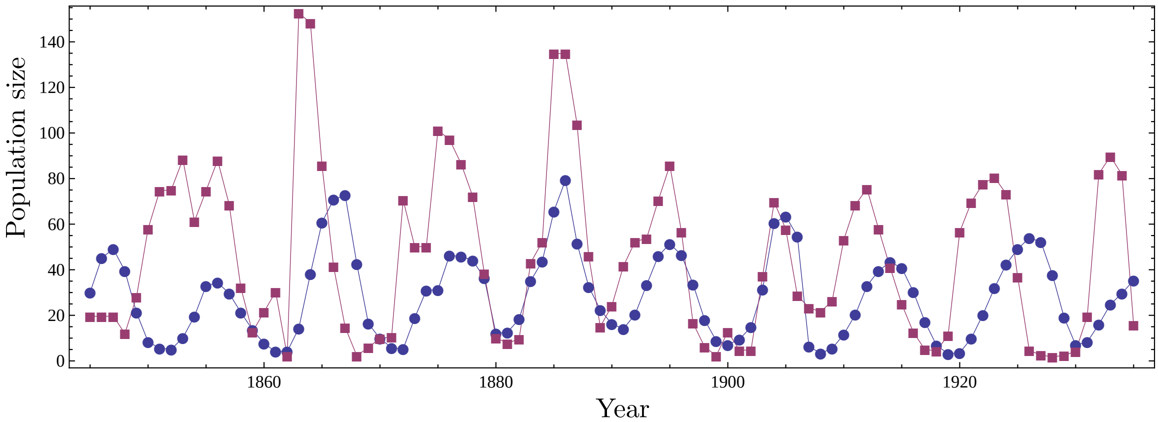

Figure 1: Data on lynx and hare in Canada from Hudson Bay Company. Circles show lynx data, and squares provide snowshoe hare data, the numbers are given in thousands

Before focusing our attention on the mathematical side of the description of the periodic phenomena, I would like to make a few remarks about these data. This figure is originally from a very respectful book by Odum "Fundamentals of Ecology." Odum says that his graph is taken from MacLulich's "Fluctuations in the numbers of varying hare,"1937, which is not widely available. Some authors caution that this data are actually a composition of several time series, and should probably not be analyzed as a whole. A great example of misinterpreting these data is given by Gilpin[1], see the figure from the cited paper. In this figure Gilpin uses data from Hudson Bay Company, which are, however, different from the data in Odum, to argue that the direction of the data change in the phase plane (hare, lynx) is clearly clockwise, whereas our simple mathematical models (and the classical Lotka–Volterra model in the first place) show counter clockwise movement, in which case the maxima of the prey population precede the maxima of the predator population. Can we discard our mathematical models on the grounds of these data? Probably not, since there are so many issues with collecting these data, including the obvious fact that these are not actual population numbers, but the number of traded pelts, which can reflect many other things. Much more on this particular example, and other examples of periodic data in biology can be found in a book by Peter Turchin[2].

Math 484/684: Mathematical modeling of biological processes by Artem Novozhilov
e-mail: artem.novozhilov@ndsu.edu. Fall 2015.

[1]Gilpin, M. E. *Do hares eat lynx?* American Naturalist (1973): 727–730.

[2]Turchin, P. (2003). Complex population dynamics: a theoretical/empirical synthesis (Vol. 35). Princeton University

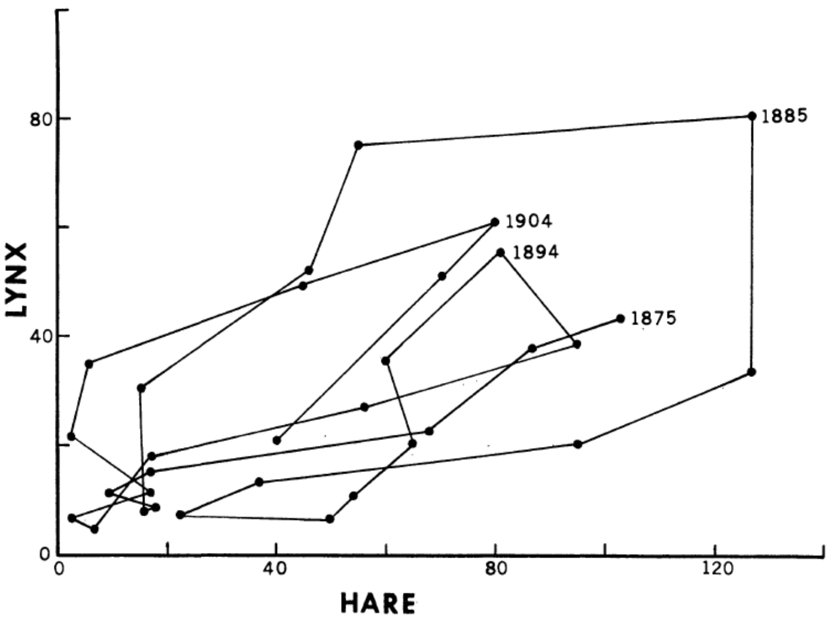

FIG. 1.—Yearly states of the Canadian lynx-hare system from 1875 to 1906. The numbers on the axes represent the numbers of the respective animals in thousands.

14.2 Limit cycles. Definitions. Stability

The simplest type of an asymptotic behavior of solutions to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in U \subseteq \mathbf{R}^2 \tag{1}$$

is arguably the equilibrium points. As I already presented in the example of the Lotka–Volterra this is not the only possibly behavior: A closed curve that corresponds to a periodic solution can represent an asymptotic behavior. I also discussed that for autonomous systems (1) any periodic solution corresponds to a closed curve in the phase space (which is quite trivial), and, in the opposite direction (which is less trivial and not true for non-autonomous system), a closed curve in the phase plane implies existence of a periodic solution, i.e., of the solution $\boldsymbol{x}(t;\boldsymbol{x}_0)$ such that

$$\boldsymbol{x}(t+T;\boldsymbol{x}_0) = \boldsymbol{x}(t;\boldsymbol{x}_0)$$

for any $t \in \mathbf{R}$, and here $T > 0$ is the minimal such real number. The examples of the closed curves were in the Lotka–Volterra model, which is *structurally unstable* (i.e., the behavior of the orbits can be destroyed by any, no matter how small, generic changes of the right hand sides of the system). Moreover, any system that possesses a family of closed curves filling a whole domain is structurally unstable. I am interested, however, in the properties of the models that persist under small changes of the equations. It turns out that a closed curve that is structurally stable has to be *isolated.* Therefore, I have the following definition.

Definition 1. *A closed orbit γ of* (1) *is called a limit cycle if it is isolated, i.e., there are no other closed curves is a small enough neighborhood of γ.*

In Figure 2 an example of a limit cycle of (1) is shown.

Press.

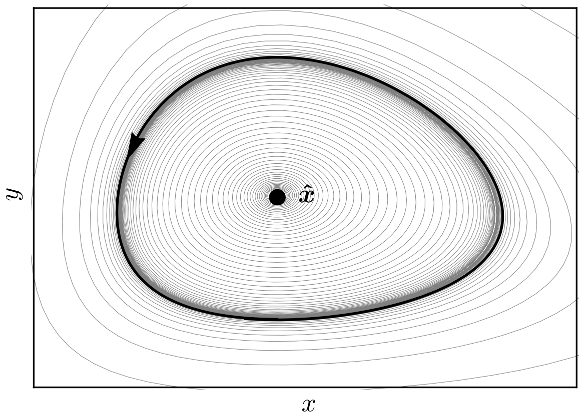

Figure 2: An asymptotically stable limit cycle

Example 2. Here is a basic example, which shows that limit cycles are stable under small system's perturbations.

Consider an ODE system in polar coordinates

$$\dot{r} = r(1 - r^2),$$
$$\dot{\theta} = 1,$$

where r is the distance to the origin, and θ is the polar angle. The equations here are decoupled and can be easily analyzed. For r I find that there are two equilibria $\hat{r}_0 = 0$ and $\hat{r}_1 = 1$ (-1 is not an equilibrium because of the definition that $r \geq 0$). The former equilibrium is unstable and the latter is asymptotically stable, and both equilibria are hyperbolic, and this implies that they will persist under small changes in the equations. Therefore, for any initial condition different from zero, $r(t) \to 1$ as $t \to \infty$. Geometrically $r = 1$ is a circle of the radius one with the center at the origin. For θ I have that it is monotonously increasing for any t. The superposition of these two movements results in the phase portrait shown in Figure 3. In case of the first equation

$$\dot{r} = -r(1 - r^2)$$

we will have an unstable limit cycle (the student is invited to make a graph).

The notions of stability and instability of the limit cycles are intuitively clear and can be formalized by using a distance function $d(A, B)$ between sets A and B. To wit, a limit cycle γ is called *stable* (or *Lyapunov stable*) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for any initial condition $\boldsymbol{x}_0$, $d(\boldsymbol{x}_0, \gamma) < \delta$, I have that $d(\boldsymbol{x}(t; \boldsymbol{x}_0), \gamma) < \epsilon$ for all $t > 0$. A limit cycle γ is called *unstable* if it is not stable. A limit cycle γ is called *asymptotically stable*, if it is stable, and additionally, $d(\boldsymbol{x}(t; \boldsymbol{x}_0), \gamma) \to 0$ as $t \to \infty$. These definitions, however, do not provide any means to determine the stability of limit cycle analytically. I will return to the notion of stability of the limit cycle in later lectures.

14.3 Criteria of absence of the limit cycles

There are no regular methods to study limit cycles of ODE. Sometimes it is possible to prove that limit cycles do not exist in some domain G. Here is one of the most useful criteria:

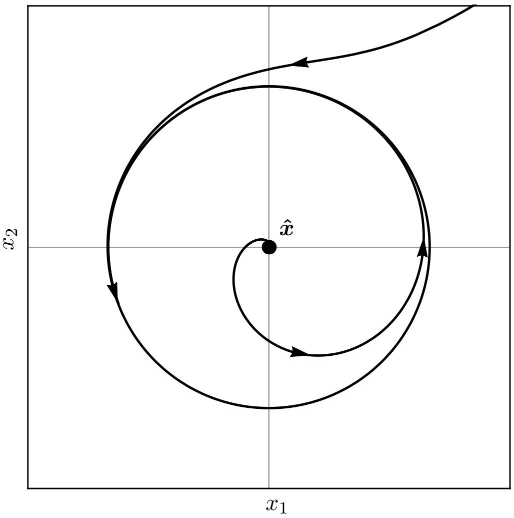

Figure 3: An asymptotically stable limit cycle in the system $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$

Proposition 3 (Dulac's criterion)**.** *Consider* (1) *and assume that* $G \subseteq U$ *is simply connected, and* $B(\boldsymbol{x}) \in \mathcal{C}^{(1)}(G; \mathbf{R})$ *such that the expression*

$$\frac{\partial}{\partial x_1}(Bf_1) + \frac{\partial}{\partial x_2}(Bf_2)$$

is sign definite (i.e., positive or negative everywhere in G*). Then there are no limit cycles in* G*.*

As a quick note, the expression $\frac{\partial}{\partial x_1}(Bf_1) + \frac{\partial}{\partial x_2}(Bf_2)$ can be concisely written as $\operatorname{div} B\boldsymbol{f}$ or $\nabla \cdot B\boldsymbol{f}$ for the del operator $\nabla = (\partial_{x_1}, \partial_{x_2})$.

Proof. Assume that there is a limit cycle $\gamma \in G$. Consider the line integral

$$I := \oint_\gamma (-Bf_2)\,\mathrm{d}x_1 + (Bf_1)\,\mathrm{d}x_1.$$

This integral, due to the assumption that γ is the orbit of (1), has to be zero:

$$I = \int_0^T \big((-Bf_2)\dot{x}_1 + (Bf_1)\dot{x}_2\big)\,\mathrm{d}t = 0.$$

On the other hand, due to Green's theorem,

$$I = \iint_D \nabla \cdot B\boldsymbol{f}\,\mathrm{d}\boldsymbol{x},$$

which cannot be zero because of the sign definiteness of the expression under the integral sign. Here D is the domain confined by γ. Therefore I arrived at a contradiction, which implies that there are no limit cycles in G. ∎

Remark 4.

- This proposition is true only for the plane, $d = 2$, and the reader is invited to think of a counterexample in dimension $d \geq 3$.
- Originally this proposition is due to Bendixson, who considered the case $B(\boldsymbol{x}) \equiv 1$. In this case the condition of the absence of the limit cycles takes the simple form that the expression

$$\nabla \cdot \boldsymbol{f}$$

is sign definite. This is why this criterion is often referred as *Bendixson–Dulac theorem.*
- A similar, but slightly more tedious proof shows that, if in G the expression $\nabla \cdot B\boldsymbol{f}$ is sign definite for some function B, then there are no simple closed curves in G, composed by the orbits. This means that not only the criterion provides the conditions for the absence of the limit cycles, but also guarantees absence of the *homo-* and/or *heteroclinic* curves composed in a closed curve.
- The condition that G is simple connected (i.e., it does not have any holes, and any two points in G can be connected) is essential. It can be shown that if G is an annular region in which $\nabla \cdot B\boldsymbol{f}$ is sign definite, then G cannot contain more than one limit cycle.

Let me use Dulac's criterion to show that the general Lotka–Volterra system on the plane cannot have limit cycles.

Example 5. The general Lotka–Volterra model

$$\begin{aligned} \dot{x} &= x(b_1 + a_{11}x + a_{12}y), \\ \dot{y} &= y(b_2 + a_{21}x + a_{22}y), \end{aligned}$$

cannot have limit cycles in $\mathbf{R}^2$ if $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

First note that the axes $x = 0$ and $y = 0$ consist of orbits, hence, the limit cycles, if exist, should lay in one of the quadrants.

Consider

$$B(x, y) = x^{\alpha-1}y^{\beta-1},$$

where α and β to be determined. I calculate

$$\nabla \cdot B\boldsymbol{f} = B(x, y)\big((\alpha a_{11} + \beta a_{21} + a_{11})x + (\alpha a_{21} + \beta a_{22} + a_{22})y + \alpha b_1 + \beta b_2\big).$$

By choosing α and β such that $\alpha a_{11} + \beta a_{21} + a_{11} = 0$ and $\alpha a_{21} + \beta a_{22} + a_{22} = 0$ (this always can be done due to the assumption), I have

$$\nabla \cdot B\boldsymbol{f} = B(x, y)(\alpha b_1 + \beta b_2).$$

If

$$\alpha b_1 + \beta b_2 \neq 0$$

then by applying Dulac's criterion I obtain the conclusion. If $\alpha b_1 + \beta b_2 = 0$ then the system admits integrating factor $B(x, y)$, can be integrated, and the nontrivial equilibrium will be surrounded by a family of the closed curves (as in the case of the classical predator–prey Lotka–Volterra model).