# Algebraic Geometry Notes

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# Contents

# Introduction

What is Algebraic Geometry?

The study of geometric objects determined by algebraic "data", i.e. polynomials. Some examples are lines in $\mathbb{R}^2$, conics in $\mathbb{R}^2$, planes in $\mathbb{R}^3$, spheres, ellipsoids, etc. in $\mathbb{R}^3$.

Geometric objects of interest include: solution sets to systems of polynomial equations, study them using algebraic techniques. For example: is the solution set finite or infinite?

**Example 0.0.1** Let $f, g \in \mathbb{C}[x, y, z]$ and let $V = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0 = g(x, y, z)\}$. Assume that $V \neq \emptyset$ (Hilbert's Nullstellensatz says that this is equivalent to $(f, g) \neq \mathbb{C}[x, y, z]$). Then $V$ is infinite. This is an application of two algebraic results: Hilbert's Nullstellensatz and Noether's Normalization Lemma.

Chapter 1

# Affine Space

## 1.1 Algebraic Subsets

Notation: Throughout these notes, $k$ will represent a field.

**Definition 1.1.1** Given an integer $n \geqslant 1$, the set $\mathbb{A}_k^n = k^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in k\}$ is a *affine $n$-space* over $k$.

**Example 1.1.2** $\mathbb{A}_{\mathbb{R}}^n = \mathbb{R}^n$ (as a set), and $\mathbb{A}_k^1 = k$ (as a set).

**Fact 1.1.3** Given $f \in k[x_1, \ldots, x_n]$ and $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{A}_k^n$, the element $f(\underline{a}) \in k$ is well-defined, i.e. $f : \mathbb{A}_k^n \to k$ is a well-defined function. These are called "regular functions." $k[x_1, \ldots, x_n]$ is the *ring of regular functions* on $\mathbb{A}_k^n$.

Note: Different polynomials can describe the same function.

**Example 1.1.4** Let $k = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$, $f(x) = x$, and $g(x) = x^p$. Fermat's Little Theorem implies that $x^p = x$ for all $x \in k$.

**Definition 1.1.5** For each $S \subseteq k[x_1, \ldots, x_n]$ set $V(S) = \{\underline{a} \in \mathbb{A}_k^n \mid f(\underline{a}) = 0 \,\forall\, f \in S\}$. $V$ is for "variety" or "vanishing." $V(S)$ is the solution set to the system of polynomial equations

$$\{f = 0 \mid f \in S\}$$

and is called *the vanishing locus* for $S$. Notation: If $S = \{f_1, \ldots, f_m\}$, we write $V(f_1, \ldots, f_m)$ instead of $V(\{f_1, \ldots, f_m\})$.

**Example 1.1.6** $V(0) = \mathbb{A}_k^n = V(\emptyset)$ and $V(1) = \emptyset = V(k[x_1, \ldots, x_n])$.

In $\mathbb{A}_{\mathbb{R}}^2$: $V(ax + by + c) =$ line, $V(x^2 + y^2 - 1) =$ circle, and similarly for other conics.

In $\mathbb{A}_{\mathbb{R}}^3$: $V(ax + by + cz + d) =$ plane, $V(ax + by + cz + d, \alpha x + \beta y + \gamma z + \delta)$ is a line as long as the two planes are distinct and non-parallel, $V(a^2x^2 + b^2y^2 + c^2z^2 - d^2) =$ ellipsoid where $a, b, c, d \neq 0$.

**Example 1.1.7** In $\mathbb{A}_k^1$: either $V(S) = \emptyset$, $V(S) = \mathbb{A}_k^1$ or $V(S)$ is finite, and for every finite set $V$ there exists $S$ such that $V = V(S)$.

**Definition 1.1.8** A subset $V \subseteq \mathbb{A}_k^n$ is an *algebraic subset* if there exists $S \subseteq k[x_1, \dots, x_n]$ such that $V = V(S)$.

**Lemma 1.1.9** *Let $S \subseteq S' \subseteq k[x_1, \dots, x_n]$. Then $V(S) \supseteq V(S')$.*

**Proof:** Exercise ■

**Proposition 1.1.10** *Let $S \subseteq k[x_1, \dots, x_n]$ and $I = (S) \subseteq k[x_1, \dots, x_n]$. Then $V(S) = V(I)$.*

**Proof:** Since $I = (S) \supseteq S$, the previous lemma implies $V(I) \subseteq V(S)$.

For the other containment, let $\underline{a} \in V(S)$. Then for all $f \in S$, $f(\underline{a}) = 0$. Therefore for all $f_1, \dots, f_m \in S$, for all $g_1, \dots, g_m \in k[x_1, \dots, x_n]$, and an arbitrary element $h = \sum_{i=1}^m g_i f_i \in I$, we have

$$h(\underline{a}) = \sum_{i=1}^{m} g_i(\underline{a}) f_i(\underline{a}) = 0.$$

Thus $\underline{a} \in V(I)$ and $V(S) = V(I)$. ■

**Notation 1.1.11** We will denote the ring $R := k[x_1, x_2, \dots, x_m]$.

**Proposition 1.1.12** *(a) For each $S_i \subset R$ and $I_i = (S_i)R$*

$$\begin{aligned} V(S_1) \cup V(S_2) \cup \cdots \cup V(S_m) &= V(I_1) \cup V(I_2) \cup \cdots \cup V(I_m) \\ &= V(I_1 \cap I_2 \cap \cdots \cap I_m) \\ &= V(I_1 I_2 \cdots I_m). \end{aligned}$$

*(b) For all $\lambda \in \Lambda$ let $S_\lambda \subseteq R$ and $I_\lambda = (S_\lambda)R$:*

$$\begin{aligned} \textstyle\bigcap_{\lambda\in\Lambda} V(S_\lambda) &= \textstyle\bigcap_{\lambda\in\Lambda} V(I_\lambda) \\ &= V\left(\textstyle\sum_{\lambda\in\Lambda} I_\lambda\right) \\ &= V\left(\textstyle\bigcup_{\lambda\in\Lambda} S_\lambda\right) \end{aligned}$$

*where $\Lambda$ is an index set that is not necessarily finite.*

*(c) The set of algebraic subsets of $\mathbb{A}_k^n$ is closed under finite unions and arbitrary intersections.*

**Proof:** (a) We first note that $V(S_1) \cup V(S_2) \cup \cdots \cup V(S_m) = V(I_1) \cup V(I_2) \cup \cdots \cup V(I_m)$ since $V(S_t) = V(I_t)$ by the previous proposition.

Next we notice that $I_j \supseteq I_1 \cap I_2 \cap \cdots \cap I_m$ for all $j$. Hence by Lemma 1.1.9 we have

$$V(I_j) \subseteq V(I_1 \cap I_2 \cap \cdots \cap I_m)$$

for all $j$. Therefore $\bigcup_j V(I_j) \subseteq V(I_1 \cap I_2 \cap \cdots \cap I_m)$.

Now since $I_1 I_2 \cdots I_m \subseteq I_1 \cap I_2 \cap \cdots \cap I_m$, by Lemma 1.1.9 we know

$$V(I_1 I_2 \cdots I_m) \supseteq V(I_1 \cap I_2 \cap \cdots \cap I_m).$$

Finally let $\underline{a} \in \mathbb{A}_k^n \setminus \bigcup_j V(I_j)$. Then we have $\underline{a} \notin V(I_j)$ for all $j$. Therefore for all $j$ there exists $f_j \in I_j$ such that $f_j(\underline{a}) \neq 0$. So let $f = f_1 f_2 \cdots f_m \in I_1 I_2 \cdots I_m$. Then

$$f(\underline{a}) = f_1(\underline{a}) f_2(\underline{a}) \cdots f_m(\underline{a}) \neq 0.$$

Therefore $\underline{a} \notin V(I_1 \cdot I_2 \cdots I_m)$. Hence what we have shown is

$$\begin{aligned} V(I_1) \cup V(I_2) \cup \cdots \cup V(I_m) &\subseteq V(I_1 \cap I_2 \cap \cdots \cap I_m) \\ &\subseteq V(I_1 I_2 \cdots I_m) \\ &\subseteq V(I_1) \cup V(I_2) \cup \cdots \cup V(I_m) \end{aligned}$$

giving us equality at each stage.

(b) Again by Lemma 1.1.9, $\bigcap_{\lambda \in \Lambda} V(S_\lambda) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$ since $V(S_\lambda) = V(I_\lambda)$ for all $\lambda \in \Lambda$.

Next we will show $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda)$. We will first show $(\supseteq)$. Here we note

$$I_\mu \subseteq \sum_{\lambda \in \Lambda} I_\lambda \text{ for all } \mu \in \Lambda.$$

Hence $V(I_\mu) \supseteq V(\sum_{\lambda \in \Lambda} I_\lambda)$ for all $\mu$ (again by Lemma 1.1.9). Therefore $\bigcap_\mu V(I_\mu) \supseteq V(\sum_{\lambda \in \Lambda} I_\lambda)$.

$(\subseteq)$ Let $\underline{a} \in \bigcap_{\lambda \in \Lambda} V(I_\lambda)$. Then $\underline{a} \in V(I_\lambda)$ for all $\lambda$. Therefore for all $f_\lambda \in I_\lambda$, $f_\lambda(\underline{a}) = 0$. So let $f \in \sum_{\lambda \in \Lambda} I_\lambda$. Then

$$f = \sum_{\lambda \in \Lambda}^{\text{finite}} f_\lambda$$

where $f_\lambda \in I_\lambda$ for all $\lambda$. Therefore

$$f(\underline{a}) = \sum_{\lambda \in \Lambda} f_\lambda = \sum_{\lambda \in \Lambda} 0 = 0.$$

Hence $\underline{a} \in V(\sum_{\lambda \in \Lambda} I_\lambda)$ as desired.

Now for the last equality we have

$$V\left(\bigcup_{\lambda \in \Lambda} S_\lambda\right) = V\left(\left(\bigcup_{\lambda \in \Lambda} S_\lambda\right) R\right) = V\left(\sum_{\lambda \in \Lambda} I_\lambda\right)$$

since $\left(\bigcup_{\lambda \in \Lambda} S_\lambda\right) R = \sum_{\lambda \in \Lambda} I_\lambda$. ■

**Definition 1.1.13** A *hypersurface* in $\mathbb{A}_k^n$ is a subset of the form $V(f)$ for a single $f$.

**Corollary 1.1.14** *Let $V$ be an algebraic subset. Then $V$ is a finite intersection of hypersurfaces.*

**Proof:** Since $V$ is algebraic and, $V = V(S) = V(I)$ where $I = (S)R$. The Hilbert Basis Theorem allows us to write $I = (f_1, f_2, \ldots, f_n)R$. Now apply $V(-)$ to get:

$$\begin{aligned} V(I) &= V(f_1, f_2, \ldots, f_m) \\ &= V(f_1 R + f_2 R + \cdots + f_m R) \\ &= V(f_1 R) \cap V(f_2 R) \cap \cdots \cap V(f_m R) \\ &= V(f_1) \cap V(f_2) \cap \cdots \cap V(f_m). \end{aligned}$$

Hence $V$ is a finite intersection of hypersufaces. ■

Notation: Given $\underline{a} \in \mathbb{A}^n_k$, let $\mathfrak{m}_{\underline{a}} := (x_1 - a_1, x_2 - a_2, \ldots, x_j - a_j, \ldots, x_n - a_n)R$.

**Fact 1.1.15** $\mathfrak{m}_{\underline{a}} \subseteq R$ is maximal because

$$\begin{aligned} \phi_{\underline{a}} &: R \to k \\ &: f \mapsto f(\underline{a}) \end{aligned}$$

is a ring epimorphism such that $\mathfrak{m}_{\underline{a}} = \mathrm{Ker}(\phi_{\underline{a}})$.

**Proposition 1.1.16** $\{\underline{a}\} = V(\mathfrak{m}_{\underline{a}}) = V(x_1 - a_2, x_2 - a_2, \ldots, x_n - a_n)$. *That is every singleton is algebraic.*

**Proof:** First notice that by the Proposition 1.1.10 we have the second equality. So we will just show $\{\underline{a}\} = V(\mathfrak{m}_{\underline{a}})$.

($\subseteq$) Since $\underline{a}$ satisfies $x_i - a_i$ for all $i$, we have $\underline{a} \in V(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$ and $\{\underline{a}\} \subseteq V(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.

($\supseteq$) If $\underline{b} \in V(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$, then $\underline{b}$ satisfies $x_i - a_i$ for all $i$. Therefore

$$\begin{aligned} b_i - a_i &= 0 \\ \Rightarrow b_i &= a_i \text{ for all } i \\ \Rightarrow \underline{b} &= \underline{a}. \end{aligned}$$

Hence $V(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \subseteq \{\underline{a}\}$. ■

**Corollary 1.1.17** *Every finite subset of $\mathbb{A}^n_k$ is algebraic.*

**Proof:** {algebraic subsets} contains all singletons and is closed under finite unions. ■

## 1.2 Zariski Topology

**Definition 1.2.1** The *Zariski Toplogy* on $\mathbb{A}^n_k$.

- A subset $V \subseteq \mathbb{A}^n_k$ is *closed* if it is algebraic.
- A subset $U \subseteq \mathbb{A}^n_k$ is *open* if $\mathbb{A}^n_k \setminus U$ is closed, i.e. algebraic. A set is open if and only if its complement is an algebraic set.

Notation: For all $f \in R$, set $U_f := \{\underline{a} \in \mathbb{A}^n_k | f(\underline{a}) \neq 0\} = \mathbb{A}^n_k \setminus V(f)$. Therefore $U_f$ is open in the Zariski Topology. These are "principal" open sets.

**Theorem 1.2.2** *The Zariski Toplogy is a topology.*

**Proof:** $\emptyset$ is algebraic and is closed in the Zariski Topology. Therefore $\emptyset^c = \mathbb{A}^n_k$ is open. Similarly $\mathbb{A}^n_k$ is algebraic, hence closed. Thus $(\mathbb{A}^n_k)^c = \emptyset$ is open.

{algebraic sets} is closed under finite unions and arbitrary intersections. Thus {open sets} is closed under finite intersections and arbitrary unions by DeMorgan's laws. ■

**Example 1.2.3** Zariski Toplogy on $\mathbb{A}^1_k$ is "cofinite topology" where

$$\{\text{open sets}\} = \{\emptyset\} \cup \{\text{complement of finite sets}\}.$$

See Example 1.1.7

**Theorem 1.2.4** *Every open set in $\mathbb{A}^n_k$ is a finite union of sets of the form $U_f$.*

**Proof:** Consider an open set $U = \mathbb{A}^n_h \setminus V(I)$. Then by Corollary 1.1.14

$$\begin{aligned} V(I) &= V(f_1) \cap V(f_2) \cap \cdots \cap V(f_m) \\ \Rightarrow U &= \mathbb{A}^n_k \setminus V(I) \\ &= (\mathbb{A}^n_k \setminus V(f_1)) \cap (\mathbb{A}^n_k \setminus V(f_2)) \cap \cdots \cap (\mathbb{A}^n_k \setminus V(f_m)) \\ &= U_{f_1} \cup U_{f_2} \cup \cdots \cup U_{f_m}. \end{aligned}$$

Thus since $U$ is an arbitrary open set, every open set is a finite union of sets of the form $U_f$. ■

**Corollary 1.2.5** *$\{U_f \mid f \in R\}$ is a basis for the Zariski topology in $\mathbb{A}^n_k$.*

Goal: Open sets are really big (when $k$ is infinite). See Theorem 1.2.8

If $k$ is finite, then every subset of $\mathbb{A}^n_k$ is finite, therefore closed and open by Corollary 1.1.17. Also, for all $\underline{a}, \underline{b} \in \mathbb{A}^n_k$ the sets $\{\underline{a}\}$ and $\{\underline{b}\}$ are open. Thus $\mathbb{A}^n_k$ is Hausdorff in this case.

**Fact 1.2.6** For all $f, g \in R$, $U_f \cap U_g = U_{fg}$.

**Proof** We note that since $k$ is a field, it is also an integral domain. Therefore $f(\underline{a}) \cdot g(\underline{a}) \neq 0$ if and only if $f(\underline{a}) \neq 0$ and $g(\underline{a}) \neq 0$. Also $f(\underline{a}) \cdot g(\underline{a}) \neq 0$ if and only if $\underline{a} \in U_{fg}$. Another thing to notice is $f(\underline{a}) \neq 0$ and $g(\underline{a}) \neq 0$ if and only if $\underline{a} \in U_f \cap U_g$. Therefore $U_{fg} = U_f \cap U_g$. ■

**Lemma 1.2.7** *Assume that $k$ is infinite and let $f, g \in R$. If $f(\underline{a}) \cdot g(\underline{a}) = 0$ for all $\underline{a} \in \mathbb{A}^n_k$, then $f \cdot g = 0$ in $R$. Therefore either $f = 0$ or $g = 0$.*

**Proof:** We will use induction on $n$. Base Case: $n = 1$. A nonzero polynomial in $k[x_1]$ can only have a finite number of zeros.

Inductive Step: Assume that $n \geqslant 2$ and that the result holds for polynomials in $k[x_2, \ldots, x_n]$. Now write

$$\begin{aligned} f &= f_0 + x_1 f_1 + x_1^2 f_2 + \cdots + x_1^d f_d \\ g &= g_0 + x_1 g_1 + x_1^2 g_2 + \cdots + x_1^e g_e \end{aligned}$$

such that all $f_i, g_i \in k[x_2, \ldots, x_n]$. Assume $f, g \neq 0$. Therefore assume $f_d \neq 0$ and $g_e \neq 0$. The induction hypothesis implies there exists $\underline{b} \in \mathbb{A}^{n-1}_k$ such that $f_d(\underline{b}) \cdot g_e(\underline{b}) \neq 0$. Hence

$$fg = f_0 g_0 + \cdots + f_d g_e x_1^{d+e} \text{ and } f(\underline{a}) \cdot g(\underline{a}) = 0$$

for all $\underline{a} \in \mathbb{A}^n_k$. Therefore $h(x_1) := f(x_1, \underline{b}) \cdot g(x_1, \underline{b}) \in k[x_1]$ where $h(c) = 0$ for all $c \in k$. Hence $h = 0$. We also have

$$h(x_1) = f_0(\underline{b}) g_0(\underline{b}) + f_1(\underline{b}) g_1(\underline{b}) x_1 + \cdots + f_d(\underline{b}) g_e(\underline{b}) x_1^{d+e} \neq 0.$$

Therefore $h \neq 0$ contradicting the above statement that $h = 0$. ■

**Theorem 1.2.8** *Assume that $k$ is infinite and let $U, U' \subseteq \mathbb{A}^n_k$ where both $U, U'$ are open and non-empty. Then $U \cap U' \neq \emptyset$.*

**Proof:** Using Theorem 1.2.4, we will write

$$U = U_{f_1} \cup U_{f_2} \cup \cdots \cup U_{f_m} \text{ and } U' = U_{g_1} \cup U_{g_2} \cup \cdots \cup U_{g_p}$$

where each $U_{f_i}, U_{g_j} \neq \emptyset$. Note that $U \cap U' \supseteq U_{f_1} \cap U_{g_1} = U_{f_1 g_1}$.

Claim: $U_{f_1 g_1} \neq \emptyset$.

If $U_{f_1 g_1} = \emptyset$, then for all $\underline{a} \in \mathbb{A}^n_k$: $f_1(\underline{a})g_1(\underline{a}) = 0$. Therefore Lemma 1.2.7 implies $f_1 = 0$ or $g_1 = 0$. But this implies that $U_{f_1} = \emptyset$ or $U_{g_1} = \emptyset$ which contradicts that $U_{f_1}, U_{g_1} \neq \emptyset$. ■

**Corollary 1.2.9** *If $k$ is infinite, then $\mathbb{A}^n_k$ is not Hausdorff.*

**Fact 1.2.10** If $\underline{a}, \underline{b} \in \mathbb{A}^n_k$ such that $\underline{a} \neq \underline{b}$, then there exists a neighborhood $U$ of $\underline{a}$ such that $\underline{b} \notin U$.

**Proof:** Let $a_i \neq b_i$ for some $i$. Then $\underline{b}$ satisfies $x_i - b_i$, but $\underline{a}$ does not. Thus $\underline{a} \in U_{x_i - b_i}$ and $\underline{b} \notin U_{x_i - b_i}$. ■

## 1.3 Geometric Ideals

**Definition 1.3.1** Given $V \subseteq \mathbb{A}^n_k$ (any subset) $I(V) = (\{f \in R \mid f(\underline{a}) = 0 \ \forall \ \underline{a} \in V\})R$. We write $I(\underline{a_1}, \underline{a_2}, \dots) := I(\{\underline{a_1}, \underline{a_2}, \dots\})$.

**Example 1.3.2** $I(\emptyset) = R$, $I(\underline{a}) = \mathfrak{m}_{\underline{a}}$. If $k$ is infinite then $I(\mathbb{A}^n_k) = 0$.

**Proposition 1.3.3** *(a) $I(V)$ is an ideal of $R$.*

*(b) If $V \subseteq V'$, then $I(V) \supseteq I(V')$.*

*(c) $I(V_1 \cup V_2 \cup \cdots \cup V_m) = I(V_1) \cap I(V_2) \cap \cdots \cap I(V_m)$.*

*(d) $I(\bigcap_{\lambda \in \Lambda} V_\lambda) \supseteq \sum_{\lambda \in \Lambda} I(V_\lambda)$.*

*(e) $\operatorname{rad}(I(V)) = I(V)$.*

**Proof:** (a) and (b) are left as exercises. (c) ($\subseteq$) We first note

$$V_1 \cup V_2 \cup \cdots \cup V_m \supseteq V_j.$$

So by part (b) we know $I(V_1 \cup V_2 \cup \cdots \cup V_m) \subseteq I(V_j)$. Therefore $I(V_1 \cup V_2 \cup \cdots \cup V_m) \subseteq \bigcap_{i=1}^m I(V_i)$.

($\supseteq$) Let $f \in I(V_1) \cap I(V_2) \cap \cdots \cap I(V_m)$. Then $f \in I(V_j)$ for all $j$. Therefore for all $\underline{a} \in V_j$, $f(\underline{a}) = 0$ for all $j$. Hence for all $\underline{a} \in V_1 \cup V_2 \cup \cdots \cup V_m$, $f(\underline{a}) = 0$. Thus

$$f \in I(V_1 \cup V_2 \cup \cdots \cup V_m).$$

(d) First note that $\bigcap_{\lambda \in \Lambda} V_\lambda \subseteq V_\mu$ for all $\mu \in \Lambda$. Therefore by part (b)

$$I\left(\bigcap_{\lambda \in \Lambda} I_\lambda\right) \supseteq I(V_\mu).$$

Therefore $I(\bigcap_{\lambda\in\Lambda} V_\lambda) \supseteq \sum_{\lambda\in\Lambda} I(V_\lambda)$.

(e) ($\supseteq$) $\mathrm{rad}(J) \supseteq J$

($\subseteq$) Let $f \in \mathrm{rad}(I(V))$. Then there exists $m$ such that $f^m \in I(V)$. Therefore for all $\underline{a} \in V$:

$$f^m(\underline{a}) = 0$$
$$(f(\underline{a}))^m = 0 \text{ in } k.$$

Therefore $f(\underline{a}) = 0$ and $f \in I(V)$. ■

**Proposition 1.3.4** *Let $I \subseteq R$ be an ideal:*

*(a)* $I \subseteq I(V(I))$.

*(b)* $V \subseteq V(I(V))$.

*(c)* $\mathrm{rad}(I) \subseteq I(V(I))$.

*(d)* $\overline{V} \subseteq V(I(V))$.

**Proof:** 1. Let $f \in I$. Then by definition for all $\underline{a} \in V(I)$, we have $f(\underline{a}) = 0$. Note that $g \in I(V(I))$ if and only if $g(\underline{a}) = 0$ for all $\underline{a} \in \mathbb{A}^n_k$. Therefore $f \in I\left(V(I)\right)$ and $I \subseteq I(V(I))$. 2. is proved similarly.

3. Let $I \subseteq I(V(I))$. Then $\mathrm{rad}(I) \subseteq \mathrm{rad}(I(V(I)) = I(V(I))$.

4. $\overline{V}$ is the closure of $V$ in $\mathbb{A}^n_k$ which is the intersection of all closed subsets of $\mathbb{A}^n_k$ containing $V$. Also $\overline{V}$ is the unique smallest closed subset of $\mathbb{A}^n_k$ containing $V$.

So $V(I(V))$ is a closed subset of $\mathbb{A}^n_k$ containing $V$ and $\overline{V}$ is the unique smallest such subset. So $\overline{V} \subseteq V(I(V))$. ■

**Proposition 1.3.5** $\overline{V} = V(I(V))$.

**Proof:** In Proposition 1.3.4 we proved that $\overline{V} \subseteq V(I(V))$. So we only need to show the reverse containment. For this, we notice that since $\overline{V}$ is a closed set we can write $\overline{V} = V(J)$. Hence

$$V \subseteq \overline{V} = V(J) \Rightarrow I(V) \supseteq I(\overline{V}) = I(V(J)) \supseteq J.$$

Therefore $V(I(V)) \subseteq V(J) = \overline{V}$. ■

**Proposition 1.3.6** $I(V(I(V))) = I(V)$.

**Proof:** ($\supseteq$) By Proposition 1.3.4 (a) we have $I(V(I(V))) \supseteq I(V)$.

($\subseteq$) By Proposition 1.3.4 (b) we have $V(I(V)) \supseteq V$. This implies $I(V(I(V))) \subseteq I(V)$. ■

**Proposition 1.3.7** $V(I(V(I))) = V(I)$.

**Proof:** Same proof as Proposition 1.3.6. ■

## 1.4 Hilbert's Nullstellensatz

**Theorem 1.4.1 (Hilbert's Nullstellensatz)** *Assume that $k = \overline{k}$:*

*(a)* $I(V(I)) = \operatorname{rad}(I)$.

*(b) The only maximal ideals of $R = k[x_1, \ldots, x_n]$ are the $\mathfrak{m}_{\underline{a}}$ with $\underline{a} \in \mathbb{A}^n_k$. i.e.*

$$\begin{aligned} \mathbb{A}^n_k &\longrightarrow \text{m-}\operatorname{Spec}(R) = \{\textit{maximal ideals of } R\} \\ \underline{a} &\longmapsto \mathfrak{m}_{\underline{a}} \end{aligned}$$

*is onto.*

**Proof:** Maybe later. ■

**Corollary 1.4.2** *Assume that $k = \overline{k}$:*

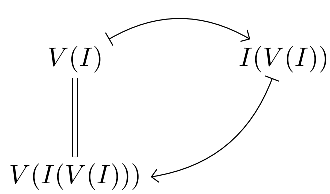
*(a)* $\{\textit{algebraic subsets of } \mathbb{A}^n_k\} \rightleftarrows \{\textit{radical ideals of } R\}$. *Under this correspondence:* $\{\underline{a}\} \rightleftarrows \mathfrak{m}_{\underline{a}}$.

*(b) If $I \neq R$, then $V(I) \neq \emptyset$.*

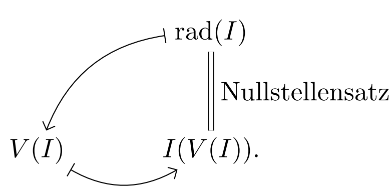
**Proof:** We know

$$\begin{aligned} V &\rightsquigarrow I(V) \\ V(I) &\leftsquigarrow I \end{aligned}$$

are well-defined. So we only need to check that they are inverses. By Proposition 1.3.7:




and for the other direction we have:




(b) If $I \neq R$, then $I(V(R)) = \operatorname{rad}(I) \neq R$. So if $V(I) = \emptyset$, then $I(V(I)) = I(\emptyset) = R$ contradicting that $I(V(R)) \neq R$. ■

**Example 1.4.3** ($k = \overline{k}$ is needed) Let $k = \mathbb{R}$ and $\mathfrak{m} = (x^2+1) \subseteq R = k[x]$. Then

$$k[x]/(x^2+1) \cong \mathbb{C}$$

which is a field. Therefore $\mathfrak{m}$ is a maximal ideal. But $I(V(\mathfrak{m})) = I(\emptyset) = R \neq \mathfrak{m} = \text{rad}(\mathfrak{m})$.

## 1.5 Irreducible Closed Subsets

**Definition 1.5.1** Let $X$ be a topological space and $V \subseteq X$ a non-empty closed subset. Then $V$ is *irreducible* if it can not be written as a union of two closed subsets properly contained in $V$.

**Example 1.5.2** $\{\underline{a}\} \subseteq \mathbb{A}^n_k$ is irreducible. For $\underline{a} \neq \underline{b}$, the set $\{\underline{a}, \underline{b}\}$ is not irreducible since

$$\{\underline{a}, \underline{b}\} = \{\underline{a}\} \cup \{\underline{b}\}.$$

**Proviso 1.5.3** If $V$ is not closed, then we need a modified definition.

**Fact 1.5.4** If $V \subseteq \mathbb{A}^n_k$ is an algebraic subset, then $V(I(V)) = V$.

**Proof:** Apply Proposition 1.3.5. ■

**Theorem 1.5.5** *Let $V \subseteq \mathbb{A}^n_k$ be a closed subset. Then $V$ is irreducible if and only if $I(V)$ is prime.*

**Proof:** ($\Rightarrow$) Assume that $V$ is irreducible. Then $V \neq \emptyset$ and hence $I(V) \neq R$. Let $f, g \in R$ such that $fg \in I(V)$. Now define

$$J := I(V) + fR$$
$$K := I(V) + gR.$$

Note that $JK \subseteq I(V) \subseteq J \cap K$ since:

$$JK = \underbrace{I(V)^2 + f \cdot I(V) + g \cdot I(V)}_{\subseteq I(V)} + \underbrace{f \cdot gR}_{\subseteq I(V)} \subseteq I(V).$$

Also note that $J = I(V) + fR \supseteq I(V)$. Similarly $K \supseteq I(V)$. Hence $J \cap K \supseteq I(V)$ and thus

$$V(JK) \supseteq V(I(V)) \supseteq V(J \cap K).$$

But $V(JK) = V(J) \cup V(K)$, $V(I(V)) = V$, and $V(J \cap K) = V(J) \cup V(K)$. Therefore

$$V(J) \cup V(K) \supseteq V \supseteq V(J) \cup V(K).$$

So $V = V(J) \cup V(K)$. Since $V$ is irreducible $V = V(J)$ or $V = V(K)$. Thus $I(V) = I(V(J)) \supseteq J$ ($f \in J$) or $I(V) = I(V(K)) \supseteq K$ ($g \in K$). So if $fg \in I(V)$ then either $f \in I(V)$ or $g \in I(V)$ and $I(V)$ is prime.

($\Leftarrow$) Assume that $I(V)$ is prime. To show $V$ is irreducible, let $V_1$ and $V_2$ be closed non-empty subsets such that $V = V_1 \cup V_2$. We need to show that $V = V_1$ or $V = V_2$. Note that for $\mathfrak{p} = I(V)$

$$V(\mathfrak{p}) = V = V_1 \cup V_2 = V(I_1) \cup V(I_2) = V(I_1 \cdot I_2).$$

So $\mathfrak{p} = I(V) = I(V(I_1 \cdot I_2)) \supseteq I_1 \cdot I_2$. Since $\mathfrak{p}$ is prime: $\mathfrak{p} \supseteq I_j$ for some $j$. Therefore

$$V_j \subseteq V_1 \cup V_2 = V = V(\mathfrak{p}) \subseteq V(I_j) = V_j.$$

Thus $V = V_j = V(I_j)$ and $V$ is irreducible. ■

**Corollary 1.5.6** *If $k = \overline{k}$ and $\mathfrak{p} \subseteq R$ is prime, then $V(\mathfrak{p})$ is irreducible.*

**Proof:** By Theorem 1.4.1 (Hilbert's Nullstellensatz) $\mathfrak{p} = I(V(\mathfrak{p}))$. Because $\mathfrak{p}$ is prime $I(V(\mathfrak{p}))$ is prime. Hence $V(\mathfrak{p})$ is irreducible by Theorem 1.5.5. ■

**Question 1.5.7** If $k$ is infinite and $\mathfrak{p} \subset R$ is prime, then $V(\mathfrak{p})$ is irreducible?

**Example 1.5.8** Let $R = \mathbb{R}[X, Y]$. Set $f = Y^2 + X^2(X-1)^2 \in R$.

Claim 1: *$(f)R \subset R$ is prime.*

Claim 2: *$V(f) = \{(0,0), (1,0)\}$ is not irreducible.*

2. Notice that $(a, b) \in V(f)$ if and only if $b = 0$ and $a(a-1) = 0$ if and only if $(a, b) = (0, 0)$ or $(1, 0)$.

1. Here we note that $R$ is a UFD. Therefore we need to prove that $f$ is irreducible. Let $f = \alpha\beta$. Since $\deg_Y(f) = 2$ we have two possibilities:

(a) $\deg_Y(\alpha) = 2$ and $\deg_Y(\beta) = 0$.

(b) $\deg_Y(\alpha) = 1$ and $\deg_Y(\beta) = 1$.

(a) $\deg_Y(\alpha) = 2$: write $\alpha = \alpha_0(X) + \alpha_1(X)Y + \alpha_2(X)Y^2$ and $\beta = \beta(X)$. Then

$$f = \alpha\beta = \beta(X)\alpha_0(X) + \beta(X)\alpha_1(X)Y + \beta(X)\alpha_2(X)Y^2.$$

So

$$\begin{aligned} \beta(X)\alpha_0(X) &= X^2(X-1)^2 \\ \alpha_1(X)\beta(X) &= 0 \\ \alpha_2(X)\beta(X) &= 1. \end{aligned}$$

The last line implies that $\beta$ is constant and hence a unit.

(b) $\deg_Y(\alpha) = \deg_Y(\beta) = 1$: write $\alpha = \alpha_0(X) + \alpha_1(X)Y$ and $\beta = \beta_0(X) + \beta_1(X)Y$. Then

$$f = \alpha\beta = \alpha_0(X)\beta_0(X) + (\alpha_0(X)\beta_1(X) + \alpha_1(X)\beta_0(X))\, Y + \alpha_1(X)\beta_1(X)Y^2$$

and so

$$\begin{aligned} \alpha_0(X)\beta_0(X) &= X^2(X-1)^2 \\ \alpha_1(X)\beta_1(X) &= 1 \\ \alpha_0(X)\beta_1(X) + \alpha_1(X)\beta_0(X) &= 0. \end{aligned}$$

The second line implies that $\alpha_1$ and $\beta_1$ are both nonzero constants. Also the third line implies that $\beta_0 = -\alpha_1^{-1}\alpha_0\beta_1$ where both $\alpha_1^{-1}$ and $\beta_1$ are constants. Now the first line implies

$$\begin{aligned} (X(X-1))^2 &= \alpha_0(X)\left(0\alpha_1^{-1}\beta_1\right)\alpha_0(X) \\ &= -\beta_1^2\alpha_0(x)^2. \end{aligned}$$

Now evaluate at $x = -1$ to see the left hand side is $> 0$ and the right hand side is $\leqslant 0$ which is a contradiction. Therefore $f$ is irreducible.

So the answer to the Question 1.5.7 is no.

**Proposition 1.5.9** *If $k$ is finite, then the only irreducible subsets of $\mathbb{A}_k^n$ are $\{\underline{a}\}$.*

**Proof:** Let $\emptyset \neq V \subseteq \mathbb{A}_k^n$ be closed such that $|V| \geqslant 2$. Note that $|V| < \infty$ because $|k| < \infty$. Also every subset of $\mathbb{A}_k^n$ is closed. So

$$V = \{\underline{a}\} \cup (V \setminus \{\underline{a}\}) \text{ for all } \underline{a} \in V.$$

So $V$ is the union of proper closed subsets. ■

**Corollary 1.5.10** *Let $k$ be finite, then $\mathbb{A}_k^n$ is reducible.*

**Proposition 1.5.11** *If $k$ is infinite, then $\mathbb{A}_k^n$ is irreducible.*

**Proof:** Suppose that $\mathbb{A}_k^n = V(I) \cup V(J)$ such that $V(I) \neq \mathbb{A}_k^n$ and $V(J) \neq \mathbb{A}_k^n$. Therefore $I \neq 0$ and $J \neq 0$. So there exists nonzero elements $f \in I$ and $g \in J$.

Claim: $\mathbb{A}_k^n = V(f) \cup V(g)$.

$f \in I$ implies that $V(f) \supseteq V(I)$. Similarly $V(g) \supseteq V(J)$. So

$$\mathbb{A}_k^n = V(I) \cup V(J) \subseteq V(f) \cup V(g) \subseteq \mathbb{A}_k^n.$$

Now for all $\underline{a} \in \mathbb{A}_k^n$: $f(\underline{a}) = 0$ or $g(\underline{a}) = 0$. Therefore for all $\underline{a} \in \mathbb{A}_k^n$:

$$f(\underline{a})g(\underline{a}) = 0.$$

If $|k| = \infty$, then $f = 0$ or $g = 0$ by Lemma 1.2.7 contradicting that $0 \neq f, g$. ■

**Corollary 1.5.12** *If $k$ is infinite, then $k^n$ can not be written as a finite union of proper subspaces.*

**Lemma 1.5.13** *Let $X$ be a topological space. The following are equivalent:*

*(i) Open sets satisfy the ascending chain condition.*

*(ii) Closed sets satisfy the descending chain condition.*

**Proof:** Exercise ■

**Definition 1.5.14** $X$ is *noetherian* if the closed sets satisfy the descending chain condition.

**Theorem 1.5.15** *$\mathbb{A}_k^n$ is noetherian.*

**Proof:** Let $V_1 \supseteq V_2 \supseteq \dots$ be a descending chain of closed subsets in $\mathbb{A}_k^n$. Now apply $I(-)$:

$$I(V_1) \subseteq I(V_2) \subseteq \dots$$

to get an ascending chain of ideals in $R = k[x_1, x_2, \dots, x_m]$. Since $R$ is noetherian the above ascending chain must stabilize. Hence $I(V_j) = I(V_{j+1}) = \cdots$ for some $j$. Note that by Fact 1.5.4

$$V_j = V(I(V_j)) = V(I(V_{j+1})) = V_{j+1}.$$

Therefore the original chain must stabilize as well. ■

**Fact 1.5.16** If $X$ is a noetherian topological space, then every closed subset of $X$ is noetherian and so is every open subset.

**Lemma 1.5.17** *Let $X$ be a topological space. The following are equivalent:*

*(i) $X$ is noetherian.*

*(ii) Every non-empty set of closed subsets of $X$ has a minimal element.*

*(iii) Every non-empty set of open subsets of $X$ has a maximal element.*

**Proof:** Exercise ■

**Theorem 1.5.18** *Let $X$ be a noetherian topological space. Then every non-empty closed subset of $X$ is a finite union of irreducible non-empty closed subsets.*

**Proof:** Suppose not. Then there exists a non-empty closed subset $V \subseteq X$ that is not a finite union of irreducible closed subsets (therefore reducible). Now define

$$\Sigma := \{\text{non-empty closed subsets of } X \text{ not a finite union of irreducible closed subsets}\}.$$

Note that $\Sigma \neq \emptyset$ is a set of closed subsets. Let $W \in \Sigma$ be a minimal element. Then $W$ is reducible. Therefore there exists closed subsets $V_1, V_2 \subseteq X$ such that $V_i \subsetneq W$ and $W = V_1 \cup V_2$. $W$ is minimal in $\Sigma$ and $V_i \subsetneq W$ is closed. Therefore $V_i \notin \Sigma$ and thus

$$V_i = V_{i,1} \cup V_{i,2} \cup \cdots \cup V_{i,m_i}$$

where $V_{i,j}$ is irreducible. $W = V_1 \cup V_2$ is a finite union of closed irreducible subsets contradicting our assumption. ■

**Corollary 1.5.19** *Every non-empty closed subset of $\mathbb{A}_k^n$ is a finite union of irreducible closed subsets.*

**Proof:** Apply Theorems 1.5.15 and 1.5.18. ■

**Theorem 1.5.20 (Uniqueness)** *Let $V_1, \ldots, V_m, V_1', \ldots, V_{m'}' \in \mathbb{A}_k^n$ be irreducible and closed such that $V_i \not\subseteq V_j$ for all $i \neq j$ and $V_i' \not\subseteq V_j'$ for all $i \neq j$ and $V_1 \cup \cdots \cup V_m = V_1' \cup \cdots \cup V_{m'}'$. Then $m = m'$ and there exists $\sigma \in S_m$ such that for all $i = 1, \ldots, m$: $V_{\sigma(i)}' = V_i$. This decomposition is called an irredundant irreducible decomposition of $V$ or a minimal irreducible decomposition of $V$.*

**Proof:** We first note that $V_1 \subseteq V_1 \cup \cdots \cup V_m = V_1' \cup \cdots \cup V_{m'}'$. Then

$$\begin{aligned} V_1 &= (V_1' \cup \cdots \cup V_{m'}') \cap V_1 \\ &= (V_1' \cap V_1) \cup \cdots \cup (V_{m'}' \cap V_1) \,. \end{aligned}$$

Since $V_1$ is irreducible and $V_i' \cap V_1$ is closed we have $V_1 = V_i' \cap V_1 \subseteq V_i'$ for some $i$. By symmetry, there exists $j$ such that $V_i' \subseteq V_j$. But no containments for the $V$'s implies $j = 1$ and hence $V_i' = V_1$. Now rearrange the $V$'s to assume $V_1 = V_1'$.

Similarly $V_2 = V'_\ell$ for some $\ell$. If $\ell = 1$, then $V_2 = V'_1 = V_1$ which is a **contradiction**. Therefore $\ell \geqslant 2$. Again we can rearrange, so assume that $V_2 = V'_2$.

For $p = 1, \dots, m$ we use a similar argument and a rearrangement will give us $V_p = V'_p$. Therefore $m \leqslant m'$. Symmetrically $m' \leqslant m$ and hence $m = m'$. Thus $V_p = V'_p$ for all $p$. ■

**Fact 1.5.21** $V \neq \emptyset$ closed in $\mathbb{A}^n_k$ implies that $V$ has an irredundant irreducible decomposition. (Remove redundancies from a given irreducible decomposition.)

**Definition 1.5.22** The $V_i$ that occur in an irredundant irreducible decomposition of $V$ are the *irreducible components* of $V$.

**Definition 1.5.23** Let $X$ be a topological space and $V \subseteq X$ be closed. The *Krull dimension* of $V$ is

$$\dim(V) := \sup\{m \geqslant 0 \mid \exists\ V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m \subseteq V \text{ such that each } V_i \text{ is closed and irreducible}\}.$$

**Definition 1.5.24** Let $S$ be a commutative ring with identity. The *Krull dimension* of $S$ is

$$\dim(S) := \sup\{m \geqslant 0 \mid \text{there exists } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m \text{ primes in } S\}$$

**Theorem 1.5.25** *Let $V \subseteq \mathbb{A}^n_k$ be closed, $V \neq \emptyset$ and $R = k[x_1, \dots, x_n]$. Then*

$$\dim(V) \leqslant \dim(R/I(V)).$$

*If $k = \overline{k}$, then equality holds.*

**Proof:** Let $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m \subseteq V$ be such that each $V_i$ is irreducible and closed. Then $I(V_0) \supseteq I(V_1) \supseteq \cdots \supseteq I(V_m) \supseteq I(V)$. Recall that $V_i$ is irreducible if and only if $I(V_i)$ is prime (Theorem 1.5.5). Note $I(V_i) \subseteq R$. So

$$\frac{I(V_0)}{I(V)} \supseteq \cdots \supseteq \frac{I(V_m)}{I(V)}$$

is a chain of prime ideals in $R/I(V)$. Then $\frac{I(V_j)}{I(V)} \supsetneq \frac{I(V_{j+i})}{I(V)}$ if and only if $I(V_j) \supsetneq I(V_{j+1})$. Suppose that $I(V_j) = I(V_{j+1})$. Then $V_j = V(I(V_j))) = V(I(V_{j+1})) = V_{j+1}$. Hence $V_j = V_{j+1}$ contradicting our assumptions. Thus

$$I(V_0) \supsetneq I(V_1) \supsetneq \cdots \supsetneq I(V_m) \supseteq I(V).$$

Therefore $\dim(R/I(V)) \geqslant m$ and thus $\dim(R/I(V)) \geqslant \dim(V)$.

For the last statement assume that $k = \overline{k}$. Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m$ be prime ideals in $R/I(V)$. Then $\mathfrak{p}_i = P_i/I(V)$ where $P_i \subseteq R$ is a prime ideal containing $I(V)$. Now the above chain of prime ideals gives us the following chain of prime ideals:

$$I(V) \subseteq P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m.$$

Therefore $V = V(I(V)) \supseteq V(P_0) \supseteq \cdots \supseteq V(P_m)$. Corollary 1.5.6 tells us $V(P_j)$ is irreducible. Now by the Nullstellensatz (Theorem 1.4.1):

$$I(V(P_j)) = \operatorname{rad}(P_j) = P_j \subsetneq P_{j+1} = I(V(P_{j+1}))$$

Therefore $V(P_j) \supsetneq V(P_{j+1})$. ■

**Example 1.5.26** If $k = \overline{k}$, then $\dim(\mathbb{A}_k^n) = \dim(k[x_1, \ldots, x_n]) = n$. If $|k| < \infty$, then $\dim(\mathbb{A}_k^n) = 0 < n$ since the only irreducible closed subsets are $\{\underline{a}\}$.

**Fact 1.5.27** If $|k| = \infty$, then $\dim(\mathbb{A}_k^n) = n$.

**Sketch of Proof:** It suffices to show $\mathbb{A}_k^m$ is irreducible for all $m$. But this is true by Proposition 1.5.11. ■

**Definition 1.5.28** $R/I(V)$ is called the *coordinate ring* for $V$. This is also known as the *ring of regular functions* on $V$. Given $\overline{f} \in R/I(V)$, then $\overline{f}$ defines a well-defined function $V \to k$.

## 1.6 Finding Irreducible Components

**Question 1.6.1** How do we find irreducible components?

**Definition 1.6.2** Let $A$ be a commutative ring with identity. An ideal $I \subsetneq A$ is *irreducible* if it can not be written as an intersection of ideals non-trivially, i.e. for all ideals $J, K \subseteq A$, if $I = J \cap K$, then $I = J$ or $I = K$.

**Example 1.6.3** If $I \subsetneq A$ is prime, then $I$ is irreducible.

**Proof:** Let $P \subset A$ be prime. Suppose that $P = J \cap K \supseteq JK$. Since $P$ is prime we have that either $P \supseteq J$ or $P \supseteq K$. Hence

$$P = \underbrace{J \cap K}_{\subseteq K} \subseteq J.$$

If $P \supseteq J$, then $J \subseteq P = J \cap K \subseteq J$, hence $P = J$. Similarly if $P \supseteq K$, then $P = K$. Therefore either $P = J$ or $P = K$. ■

**Proposition 1.6.4** *Let $A$ be a noetherian ring and let $I \subset A$ be irreducible. Then for all $x, y \in A$ if $xy \in I$, then either $x \in I$ or there exists $m \geqslant 1$ such that $y^m \in I$. (This says that $I$ is "primary").*

**Proof:** Assume that $I$ is irreducible and let $x, y \in A$ such that $xy \in I$. Now consider the colon ideals

$$(I : y) \subseteq (I : y^2) \subseteq (I : y^3) \subseteq \cdots.$$

Since $A$ is noetherian, the above chain must stabilize. Therefore for some $m \geqslant 1$ we must have $(I : y^m) = (I : y^{m+1})$.

Claim: $(I + xA) \cap (I + y^m A) = I$.

($\supseteq$) Note that $I + xA \supseteq I$ and $I + y^m A \supseteq I$. Therefore $(I + xA) \cap (I + y^m A) \supseteq I$.

($\subseteq$) Let $\alpha \in (I + xA) \cap (I + y^m)A)$. Then we can write

$$\alpha = i + xr \quad \text{for } i \in I, r \in A \tag{1.1}$$
$$\alpha = j + y^m s \quad \text{for } j \in I, s \in A. \tag{1.2}$$

Multiplying (1.1) by $y$ we see $\alpha y = iy + xry \in I$ since $\alpha y \in I$ and $xry \in I$ because $xy \in I$.

Multiplying (1.2) by $y$ we see $\alpha y = jy + y^m sy = jy + y^{m+1}s$. Note that $y^{m+1}s = \alpha y - jy \in I$. Therefore $s \in (I : y^{m+1}) = (I : y^m)$ and hence $\alpha = j + y^m s \in I$. Thus $(I + xA) \cap (I + y^m A) \subseteq I$.

Now since $I$ is irreducible, we have $I = I + xA \ni x$ or $I = I + y^m A \ni y^m$. ■

**Proposition 1.6.5** *Let $A$ be noetherian and $I \subseteq A$ be an ideal. Then $I$ is prime if and only if $I$ is irreducible and $I = \mathrm{rad}(I)$.*

**Proof:** ($\Rightarrow$) Let $I$ be a prime ideal. Then by Example 1.6.3 we have that $I$ is irreducible. Now, since $I$ is prime we also have that $I = \mathrm{rad}(I)$.

($\Leftarrow$) Assume that $I = \mathrm{rad}(I)$ is irreducible. To show that $I$ is prime, let $x, y \in A$ such that $xy \in I$. Since $I$ is irreducible Proposition 1.6.4 implies that $x \in I$ or $y^m \in I$ for some $m$. But since $I = \mathrm{rad}(I)$ it follows that $x \in I$ or $y \in I$. Hence $I$ is prime. ■

**Proposition 1.6.6** *Let $A$ be noetherian and $I \subset A$ be an irreducible ideal. Then $\mathrm{rad}(I)$ is prime.*

**Proof:** Since $I \neq A$ we have $\mathrm{rad}(I) \neq A$. Let $x, y \in A$ such that $xy \in \mathrm{rad}(I)$. Then $x^m y^m \in I$ for some $m$. Proposition 1.6.5 implies that $x^m \in I$ or $y^{ml} \in I$. Therefore $x \in \mathrm{rad}(I)$ or $y \in \mathrm{rad}(I)$. Hence $\mathrm{rad}(I)$ is prime. ■

**Fact 1.6.7** If $I \subseteq A$ is an irreducible ideal and $I = J_1 \cap J_2 \cap \cdots \cap J_m$, then $I = J_i$ for some $i$.

**Proof:** Use induction on $m$ along with the definition of $I$ being irreducible. ■

**Definition 1.6.8** A *irreducible decomposition* of an ideal is a decomposition $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ such that each $Q_i$ is irreducible. Such a decomposition is *irreduntant* if it has no redundancies: if $i \neq j$, then $Q_i \not\subseteq Q_j$.

**Proposition 1.6.9** *If $A$ is noetherian and $I \subseteq A$ is an ideal, then $I$ has an irredundant irreducible decomposition. Moreover if the $Q_i$'s are prime, then the $Q_i$'s in the irreducible decomposition are unique (up to order).*

**Proof:** Step 1: Every ideal in $A$ has an irreducible decomposition.

Assume there exists an ideal in $A$ that does not have an irreducible decomposition. Then let

$$\Sigma = \{\text{ideals } I \text{ that do not have an irreducible decomposition}\}.$$

Then $\Sigma \neq 0$ is a collection of ideals. Since $A$ is noetherian, $\Sigma$ has a maximal element $I$. Then $I$ is not irreducible (if $I$ were irreducible, then $I = I$ is an irreducible decomposition). Therefore there exists ideals $J, K$ such that $I = J \cap K$ and $I \neq J$ and $I \neq K$. Then $I \subsetneq J$ and $I \subsetneq K$. Since $I$ is maximal in $\Sigma$ we have $J, K \notin \Sigma$. Then we can write

$$J = Q_1 \cap \cdots \cap Q_m$$
$$K = L_1 \cap \cdots \cap L_t$$

as irreducible decompositions. This implies that

$$I = J \cap K = Q_1 \cdots \cap Q_m \cap L_1 \cap \cdots \cap L_t$$

is an irreducible decomposition contradicting that $I$ does not have an irreducible decomposition.

Step 2: If $I = Q_1 \cap \cdots \cap Q_m$ is an irreducible decomposition, then the $Q_i$'s can be reordered so that there exists $t$ such that

$$I = Q_1 \cap \cdots \cap Q_t$$

is irredundant.

If $Q_1 \cap \cdots \cap Q_m$ is irredundant, then we are done. If not, then there exists $i \neq j$ such that $Q_i \subseteq Q_j$. Reorder to assume $Q_i \subseteq Q_m$. Then

$$I = Q_1 \cap \cdots \cap Q_{m-1} \cap Q_m = Q_1 \cap \cdots \cap Q_{m-1}.$$

This process terminates in a finite number of steps because $m < \infty$.

Step 3: Uniqueness when irredundant and each $Q_i$ is prime.

Suppose that

$$I = Q_1 \cap \cdots \cap Q_m = L_1 \cap \cdots \cap L_t$$

are irredundant decompositions such that all $Q_i$ and $L_i$ are prime. Then

$$L_1 \cap \cdots \cap L_t = Q_1 \cap \cdots \cap Q_m \subseteq Q_1.$$

Since $Q_1$ is prime there exists $j$ such that $Q_1 \supseteq L_j$. Similarly there exists $i$ such that $L_j \supseteq Q_i$ because $L_j$ is prime. Note that the decomposition $Q_1 \cap \cdots \cap Q_m$ is irredundant, so we must have $i = 1$ and hence $Q_1 = L_j$. Now reorder the $L$'s to assume that $Q_1 = L_1$. Similarly there exists $s$ such that $Q_2 = L_s$. Note that $s \neq 1$ because then $Q_2 = L_1 = Q_1$ which is a redundancy. Therefore assume $s = 2$. Similarly reorder the $L$'s to get $L_i = Q_i$ for $i = 1, \ldots, m$. This implies that $m \leqslant t$. By symmetry $t \leqslant m$. ∎

We note that the $Q_i$'s must be prime in order to have uniqueness for irredundant decompositions. Consider the following example.

**Example 1.6.10** Consider the ideal $(x^2, xy^2) \in k[x, y]$. Note that this is an ideal such that

$$(y + x, x^2) \cap (x, (y + x)^2) = (x^2, xy^2) = (y, x^2) \cap (x, y^2).$$

Note that both decompositions are irredundant, but $(y + x, x^2) \neq (y, x^2) \neq (x, (y + x)^2)$ and $(y + x, x^2) \neq (x, y^2) \neq (x, (y + x)^2)$. So uniqueness fails.

**Proposition 1.6.11** *Let $A$ be a commutative ring with identity and let $I \subseteq A$ be an ideal. Then*

*(a) $I = \mathrm{rad}(I)$ if and only if $I = P_1 \cap \cdots \cap P_m$ for some primes $P_i$.*

*(b) If $I = \mathrm{rad}(I)$, then $I$ has a unique irredundant prime decomposition (up to reordering).*

**Proof:** (a) ($\Rightarrow$) Let $I = Q_1 \cap \cdots \cap Q_m$ be an irreducible decomposition. Then

$$I = \mathrm{rad}(I) = \mathrm{rad}(Q_1 \cap \cdots \cap Q_m) = \mathrm{rad}(Q_1) \cap \cdots \cap \mathrm{rad}(Q_m).$$

Then by Proposition 1.6.6 each $\mathrm{rad}(Q_i)$ is prime.

($\Leftarrow$) Let $I = P_1 \cap \cdots \cap P_m$ where $P_i$ is prime. Then

$$\begin{aligned} \operatorname{rad}(I) &= \operatorname{rad}(P_1 \cap \cdots \cap P_m) \\ &= \operatorname{rad}(P_1) \cap \cdots \cap \operatorname{rad}(P_m) \\ &= P_1 \cap \cdots \cap P_m \\ &= I. \end{aligned}$$

(b) Part (a) implies that $I$ has a prime decomposition. Remove the redundancies to get an irredundant prime decomposition. Then Proposition 1.6.9 implies the decomposition is unique. ∎

Now: How do we find irreducible components of $V = V(I)$?

**Proposition 1.6.12** *Assume $k = \overline{k}$, and let $I = \operatorname{rad}(I) = P_1 \cap \cdots \cap P_m$ be an irredundant prime decomposition. Then the irreducible components of $V(I) \subseteq \mathbb{A}_k^n$ are $V(P_1), \ldots, V(P_m)$.*

**Proof:** We first have

$$V(I) = V(P_1 \cap \cdots \cap P_m) = V(P_1) \cup \cdots \cup V(P_m).$$

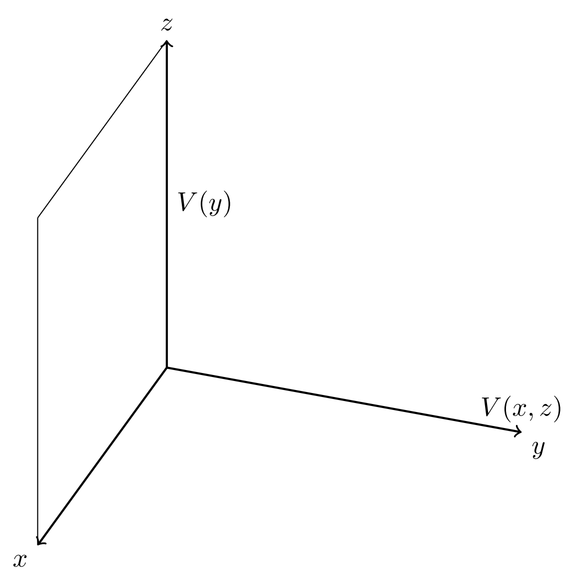
Then Theorem 1.4.1 (Hilbert's Nullstellensatz) implies that $I(V(P_i)) = \operatorname{rad}(P_i) = P_i$ for all $i$. Therefore $V(P_i)$ is irreducible for all $i$.

Next we check that this decomposition is irredundant. If $V(P_i) \subseteq V(P_j)$ for some $i \neq j$, then

$$\underbrace{I(V(P_i))}_{=P_i} \supseteq \underbrace{I(V(P_j))}_{=P_j}.$$

But this contradicts the original irredundancy. ∎

**Example 1.6.13** Assume $k = \overline{k}$ and consider $V(xy, yz) \subseteq \mathbb{A}_k^3$. Then $I = (xy, yz)R = (x, z)R \cap (y)R$ is an irredundant prime decomposition. Therefore the components of $V(I)$ are $V(x, z)$ and $V(y)$.



**Proposition 1.6.14** *Let $k = \overline{k}$ and $I \subseteq R$ (not necessarily $= \operatorname{rad}(I)$). Let $I = Q_1 \cap \cdots \cap Q_m$ be an irreducible decomposition, and consider*

$$\operatorname{rad}(I) = \operatorname{rad}(Q_1) \cap \cdots \cap \operatorname{rad}(Q_m) = P_1 \cap \cdots \cap P_t.$$

*In the last expression, we have removed the redundancies and reordered the ideals, and we have $t \leqslant m$. Then the irreducible components of $V(I) = V(\operatorname{rad}(I))$ are $V(P_1), \ldots, V(P_t)$.*

**Proof:** Apply Proposition 1.6.12. ■

**Fact 1.6.15** In $R = k[x_1, \ldots, x_n]$ each ideal $\left(x_{i_1}^{e_{i_1}}, \ldots, x_{i_m}^{e_{i_m}}\right) R = J$ where $e_{i_j} \geqslant 1$ is irreducible such that $\operatorname{rad}(J) = (x_{i_1}, \ldots, x_{i_m}) R$.

**Example 1.6.16** Assume $k = \overline{k}$ and consider $I = (x^2, y) \cap (x, z^3) \cap (y^2, z^4)$.

$$\begin{aligned} \operatorname{rad}(I) &= \operatorname{rad}(x^2, y)R \cap \operatorname{rad}(x, z^3) \cap \operatorname{rad}(y^2, z^4) \\ &= (x, y)R \cap (x, z)R \cap (y, z)R. \end{aligned}$$

The irreducible components of $V(I)$ are $V(x, y)$, $V(x, z)$, and $V(y, z)$, i.e., the $z$-axis, the $y$-axis, and the $x$-axis, respectfully.

Now what if $k \neq \overline{k}$?

**Example 1.6.17** Let $V(x^2 + y^2(y-1)^2) = \{(0,0), (0,1)\} = \{(0,0)\} \cup \{(0,1)\}$. The irreducible components are $\{(0,0)\}$ and $\{(0,1)\}$. Then if $I = (x^2 + y^2(y-1)^2)R$, the prime decomposition of $I$ does not give irreducible components of $V(I)$. We need to decompose $I(V(I))$.

**Proposition 1.6.18** *Let $I(V(I)) = P_1 \cap \cdots \cap P_m$ be an irredundant prime decomposition. Then the irreducible components of $V(I)$ are $V(P_1), \ldots, V(P_m)$.*

**Proof:** Let $V(I) = V_1 \cup \cdots \cup V_t$ be an irredundant irreducible decomposition. We need to show that $t = m$ and the $V_i$'s can be reordered to get $V_i = V(P_i)$ for all $i$. Note that

$$P_1 \cap \cdots \cap P_m = I(V(I)) = I(V_1) \cap \cdots \cap I(V_t).$$

Since $V_i$ is irreducible, we have $I(V_i)$ is prime for all $i$. Now we need to show for $i \neq j$ that $I(V_i) \not\subseteq I(V_j)$. (Then each decomposition is irredundant. Therefore uniqueness kicks in.)

If $I(V_i) \subseteq I(V_j)$, then $V(I(V_i)) \supseteq V(I(V_j))$. But $V(I(V_i)) = \overline{V}_i = V_i$ and $V(I(V_j)) = V_j$. Therefore $V_i \supseteq V_j$ which contradicts the irredundancy of $V_1 \cup \cdots \cup V_t$. Hence $I(V_1) \cup \cdots \cup I(V_t)$ is irredundant.

The uniqueness implies that $m = t$ and the $V_i$'s can be reordered to get $P_i = I(V_i)$ for all $i$. Therefore $V(P_i) = V(I(V_i)) = \overline{V}_i = V_i$ for all $i$. ■

**Example 1.6.19** Let $k = \mathbb{R}$. Then $V(x^2 + y^2(y-1)^2) = \{(0,0), (0,1)\}$ and

$$\begin{aligned} I(V(x^2 + y^2(y-1)^2)) &= I(\{(0,0), (0,1)\}) \\ &= \mathfrak{m}_{(0,0)} \cap \mathfrak{m}_{(0,1)} \\ &= (x, y)R \cap (x, y-1)R. \end{aligned}$$

So the irreducible components are $V(x, y) = \{(0,0)\}$ and $V(x, y-1) = \{(0,1)\}$.

**Exercises.**

Assumptions: $k$ is a field, and $R = k[X_1, \ldots, X_n]$ for some $n \geqslant 1$.

**Exercise 1.6.20** Let $I \subseteq R$ be an ideal, and consider the "radical" of $I$:

$$\operatorname{rad} I = \{x \in R \mid x^n \in I \text{ for some } n \geqslant 0\}.$$

Note that $\operatorname{rad} I$ is an ideal of $R$ containing $I$. Prove that $V(I) = V(\operatorname{rad} I)$.

**Exercise 1.6.21** Let $S, S' \subseteq R$.

(a) Prove that if $S \subseteq S'$, then $V(S) \supseteq V(S')$.

(b) Prove or give a counterexample to the following: if $V(S) \supseteq V(S')$, then $(S)R \subseteq (S')R$.

**Exercise 1.6.22** For each $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{A}^n_k$, set $\mathfrak{m}_{\underline{a}} = (X_1 - a_1, \ldots, X_n - a_n)R$. Let $S \subseteq R$, and prove that $\underline{a} \in V(S)$ if and only if $\mathfrak{m}_{\underline{a}} \supseteq S$.

**Exercise 1.6.23** Let $m \geqslant 1$. A function $F \colon \mathbb{A}^b_k \to \mathbb{A}^m_k$ is *regular* if there are polynomials $f_1, \ldots, f_m \in R$ such that $F(\underline{a}) = (f_1(\underline{a}), \ldots, f_m(\underline{a}))$ for all $\underline{a} \in \mathbb{A}^b_k$.

(a) Prove that every regular function $F \colon \mathbb{A}^n_k \to \mathbb{A}^m_k$ is continuous.

(b) Prove that if $F \colon \mathbb{A}^n_k \to \mathbb{A}^m_k$ and $G \colon \mathbb{A}^p_k \to \mathbb{A}^n_k$ are regular, then so is the composition $F \circ G \colon \mathbb{A}^p_k \to \mathbb{A}^m_k$.

**Exercise 1.6.24** Assume that $k$ is algebraically closed. Let $f$ be a non-constant polynomial in $k[X_1, \ldots, X_n]$, and consider $V(f) \subset \mathbb{A}^n_k$.

(a) Prove that $V(f) \neq \emptyset$.

(b) Prove that if $n \geqslant 2$, then $V(f)$ is infinite.

**Exercise 1.6.25** Are the following closed sets irreducible or not? Justify your responses.

(a) $V(X + Y^2) \subseteq \mathbb{A}^2_{\mathbb{R}}$.

(b) $V(X^2 + Y^2) \subseteq \mathbb{A}^2_{\mathbb{R}}$.

(c) $V(X^2 + Y^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$.

**Exercise 1.6.26** Let $X$ be a noetherian topological space. Let $Y \subseteq X$ be a subspace of $X$, that is, a subset of $Y$ with the subspace topology. Prove that $Y$ is noetherian.

Chapter 2

# Projective Space

## 2.1 Motivation

Sometimes to describe an object, we "parametrize" it in terms of something we already understand.

**Example 2.1.1** A curve in $\mathbb{R}^3$, described in parametric form:

$$\vec{f}(t) = (x(t), y(t), z(t)) .$$

**Example 2.1.2** Solution sets to under-determined systems of linear equations:

$$\begin{aligned} 2x + 3y + 4z &= 0 \\ x + y - z &= 0. \end{aligned}$$

Solutions: $(-\frac{7}{6}t, t, -\frac{1}{6}t)$.

General Principle: To understand $T$, cook up a function $f : S \to T$ where $f$ is "nice" and $S$ is "understood." Then transfer understanding of $S$ to $T$ via $f$.

Motivating Problem: Parametrize the set of lines in $k^n$ passing through the origin, i.e. 1-dimensional vector subspaces of $k^n$.

A line is determined by a single non-zero vector $\vec{v} \in k^n$.

$$\begin{aligned} \mathrm{Span} : k^n \setminus \{0\} &\to \{\text{lines in } k^n \text{ passing through the origin}\} \\ \vec{v} &\mapsto \mathrm{Span}(\vec{v}). \end{aligned}$$

This is onto. Sadly it is not one-to-one. Note that $\mathrm{Span}(\vec{v}) = \mathrm{Span}(\vec{w})$ if and only if $\vec{v} \in \mathrm{Span}(\vec{w})$ if and only if $\vec{w} \in \mathrm{Span}(\vec{v})$ if and only if there exists $\lambda \in k^\times$ such that $\vec{v} = \lambda\vec{w}$ if and only if there exists $\mu \in k^\times$ such that $\vec{w} = \mu\vec{v}$.

We will define $\vec{v} \sim \vec{w}$ if there exists $\lambda \in k^\times$ such that $\vec{v} = \lambda\vec{w}$. By the above notes, the induced function

$$\begin{aligned} \overline{\mathrm{Span}} : \frac{k^n \setminus \{0\}}{\sim} &\to \mathbb{P}^n_k \\ [\vec{v}] &\mapsto \mathrm{Span}(\vec{v}) \end{aligned}$$

is well-defined, one-to-one and onto.

## 2.2 Projective Space $\mathbb{P}^n_k$

**Definition 2.2.1** For $\vec{v}, \vec{w} \in k^{n+1} \setminus \{\vec{0}\}$ define $\vec{v} \sim \vec{w}$ if and only if there exists $\lambda \in k^\times$ such that $\vec{w} = \lambda \cdot \vec{v}$. If $\vec{v} = (v_0, v_1, \ldots, v_n)$ then $\vec{v} \sim \vec{w}$ if and only if there exists $\lambda \in k^\times$ such that $w_j = \lambda \cdots v_j$ for all $j = 0, 1, \ldots, n$.

$$\mathbb{P}^n_k := \frac{k^{n+1} \setminus \{0\}}{\sim}$$

is called *projective $n$-space* over $k$. An element in $\mathbb{P}^n_k$ looks like

$$\underline{v} = (v_0 : v_1 : \cdots : v_n) = (\lambda v_0 : \lambda v_1 : \cdots : \lambda v_n)$$

for all $\lambda \in k^\times$.

**Example 2.2.2** $(v_0 : v_1) \in \mathbb{P}^1_k$. If $v_0 \neq 0$ then $(v_0 : v_1) = (v_0/v_0 : v_1/v_0) = (1 : v_1')$ where $\lambda = 1/v_0$ and $v_1' = v_1/v_0$.

For $\alpha, \beta \in k$: $\alpha \neq \beta$ implies that $(1 : \alpha) \neq (1 : \beta)$. So

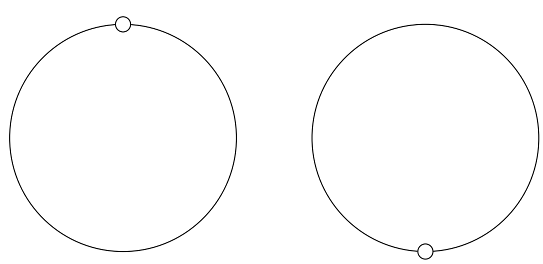
$$\begin{aligned} \mathbb{A}^1_k &\hookrightarrow \mathbb{P}^1_k \\ \alpha &\mapsto (1 : \alpha). \end{aligned}$$

If $v_0 = 0$, then $v_1 \neq 0$ and $(v_0 : v_1) = (0 : v_1) = (0/v_1 : v_1/v_1) = (0 : 1)$.

Symmetrically we have another

$$\begin{aligned} \mathbb{A}^1_k &\hookrightarrow \mathbb{P}^1_k \\ \beta &\mapsto (\beta : 1). \end{aligned}$$

Here we see that $\mathbb{P}^1_k \approx$ a circle in $\mathbb{R}^2$.



More Generally: In $\mathbb{P}^n_k$, $U_j = \{\underline{v} \in \mathbb{P}^n_k \mid v_j \neq 0\}$ for all $j = 0, 1, \ldots, n$. Then

$$\mathbb{P}^n_k = U_0 \cup U_1 \cup \cdots \cup U_n.$$

There also exists a bijection $\mathbb{A}^n_k \to U_j$ for all $j$. If $n \geqslant 2$, then there exists a bijection $\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k \setminus U_j$ for all $j$.

Note that polynomials do <u>not</u> give well-defined functions on $\mathbb{P}^n_k$.

**Example 2.2.3** $f(x_0, x_1) = x_0$. $(1 : 0) \xrightarrow{f} 1$ and $(\alpha : 0) \xrightarrow{f} \alpha$.

Being zero or not zero (evaluating polynomial) is also <u>not</u> well-defined. For $\alpha \neq 0$:

$$\begin{aligned} g(x_0, x_1) &= x_0^2 - x_1^3 \\ g(1:1) &= 1^2 - 1^3 = 0 \\ g(\alpha:\alpha) &= \alpha^2 - \alpha^3 = \alpha^2(1-\alpha) = 0 \Leftrightarrow \alpha = 1. \end{aligned}$$

We will see that homogeneous polynomials will fix our problem.

## 2.3 Homogeneous Polynomials

**Definition 2.3.1** Let $R = k[x_0, x_1, \ldots, x_n]$ and let $f \in R$. Then $f$ is *homogeneous* of degree $d$ if $f$ is a linear combination over $k$ of monomials of degree $d$.

**Example 2.3.2** $x_0^2 - x_1^3$ is not homogeneous. $x_0^3 - x_1 x_2^2$ is a homogeneous polynomial of degree 3.

**Notation 2.3.3** $R_d := \{\text{homogeneous polynomials } f \in R \text{ of degree } d\}$.

<u>Note</u>: 0 is homogeneous of degree $d$ for all $d$.

**Fact 2.3.4** $R_d \subseteq R$ is a subspace over $k$.

**Fact 2.3.5**

(a) Every $f \in k[x_0, \ldots, x_n] = R$ can be written as $f = f_0 + f_1 + \cdots + f_d$ such that each $f_i \in k[x_0, \ldots, x_n]_i = R_i$, i.e, each $f_i$ is homogeneous of degree $i$. This representation is essentially unique.

(b) Therefore $R \cong R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ .

(c) $R_i \times R_j \longrightarrow R_{i+j}$ by $(f, g) \longmapsto fg$ is a well-defined $k$-bilinear map.

(d) $f \in R_i$: $f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^i f(x_0, \ldots, x_n)$. Note that homogeneous is crucial.

(e) Let $\underline{v} = \underline{w} \in \mathbb{P}_k^n$ where $\underline{v} = (v_0 : v_1 : \cdots : v_n)$ and $\underline{w} = (w_0 : w_1 : \cdots : w_n)$. Note that $\vec{v} = \langle v_0, v_1, \ldots, v_n \rangle$ and $\vec{w} = \langle w_0, w_1, \ldots, w_n \rangle$. Then $f(\vec{v}) = 0$ if and only if $f(\vec{w}) = 0$ for $f \in R_i$.

**Proof of (e):** Assume that $\underline{v} = \underline{w}$, then $\vec{w} = \lambda \vec{v}$ for some $\lambda \in k^\times$. Therefore

$$f(\vec{w}) = f(\lambda \vec{v}) = \lambda^i f(\vec{v}).$$

Hence $f(\vec{w}) = 0$ if and only if $\lambda^i f(\vec{v}) = 0$ if and only if $f(\vec{v}) = 0$ since $\lambda \neq 0$. ■

**Notation 2.3.6** We write $f(\underline{v}) = 0$ if $f(\vec{v}) = 0$. (This is well-defined by Fact 5.)

<u>Note</u>: $f$ does not make a well-defined function $\mathbb{P}_k^n \to k$.

**Proposition 2.3.7** *Let $I \subseteq R$ be an ideal. The following are equivalent:*

*(i) $I$ is generated by a set of homogeneous polynomials.*

*(ii) $I$ is generated by a finite set of homogeneous polynomials.*

*(iii) $I = (I \cap R_0) \oplus (I \cap R_1) \oplus \cdots$ .*

*(iv) Every $f \in I$ can be written as $f = f_0 + \cdots + f_d$ such that each $f_i$ is in $I \cap R_i$.*

*(v) For all $f \in I$, if $f = \sum_{i=0}^{d} f_i$ such that each $f_i \in R_i$, then $f_i \in I$ for all $i$.*

**Definition 2.3.8** If $I \subseteq R$ is an ideal, then $I$ is a *homogeneous ideal* if it satisfies the equivalent conditions of Proposition 2.3.7.

**Proof of Proposition 2.3.7:** (i) $\Rightarrow$ (ii): The Hilbert Basis Theorem implies that $I$ is finitely generated. If $I = (S)R$ such that $S$ is a set of homogeneous polynomials, then since $I$ is finitely generated there exists a finite subset $S' \subseteq S$ such that $I = (S')R$.

(ii) $\Rightarrow$ (iv): Let $s_1, \ldots, s_m \in R$ be homogeneous such that $I = (s_1, \ldots, s_m)R$. Therefore $s_i \in I$ for all $i$. Say $s_i \in R_{d_i}$. Let $f \in I$. Then there exists $g_1, \ldots, g_m \in R$ such that $f = \sum_i g_i s_i$ where $g_i = \sum_j g_{i,j}$ such that $g_{i,j} \in R_j$. Then

$$f = \sum_i g_{i,j} s_i \sum_i \sum_j g_{i,j} s_i = \sum_t \underbrace{\sum_{j+d_i} g_{i,j} s_i}_{f_t \in R_t} .$$

This satisfies the desired conclusion because each $f_i$ is in $(s_1, \ldots, s_m)R = I$.

(iv) $\Rightarrow$ (iii): We first need to show that the sum is direct. So we need to show that $(I \cap R_i) \cap \left(\sum_{j \neq i}(I \cap R_j)\right) = 0$. Note that 0 is in the left hand side. For the other containment:

$$(I \cap R_i) \cap \left(\textstyle\sum_{j \neq i} I \cap R_j\right) \subseteq R_i \cap \textstyle\sum_{j \neq i} R_i = 0$$

since $R \cong R_0 \oplus R_1 \oplus \cdots$. Therefore the sum is direct.

($\subseteq$) Let $f \in I$. Then (iv) implies for $f_i \in R_i$:

$$f = \underbrace{f_0 + f_1 + \cdots + f_d}_{\in (I \cap R_0) \oplus (I \cap R_1) \oplus \cdots} .$$

($\supseteq$) Note that for each $R_j$, $I \cap R_j \subseteq I$.

(iii) $\Rightarrow$ (v): Assume (iii) and let $f \in I$. Then $f = \sum_i f_i$ such that $f_i \in R_i$ for all $i$. We need to show $f_i \in I$ for all $i$. Condition (iii) implies $f = \sum_i \widetilde{f_i}$ such that $\widetilde{f} \in I \cap R_i$. Note that by the uniqueness of representation in $R = R_0 \oplus R_1 \cdots$, the condition $f = \sum_i f_i$ implies $f_i = \widetilde{f_i}$ for all $i$. Therefore $f_i \in I$ for all $i$.

(v) $\Rightarrow$ (i): Assume (v) and let $S = \bigcup_{i=1}^{\infty} (I \cap R_i) = \{\text{homogeneous } f \in I\} \subseteq I$. Therefore $(S)R \subseteq I$.

($\supseteq$) Let $f \in I$ such that $f = f_0 + \cdots f_d$ where each $f_i \in R_i$. (v) implies that each $f_i \in I$. Thus $f_i \in I \cap R_i$ for all $i$. Therefore $f =$ a sum of elements of $S$. Hence $f \in (S)R$. ∎

**Example 2.3.9** If $f_1, \ldots, f_m$ are homogeneous, then $I = (f_1, \ldots, f_m)R$ is a homogeneous ideal.

**Notation 2.3.10** Let $I \subseteq R$ be homogeneous. For all $d$ let

$$I_d := \{\text{homogeneous polynomials in } I \text{ of degree } d\}.$$

**Fact 2.3.11** $I_d = I \cap R_d$.

**Lemma 2.3.12** *Let $I, J \subseteq R$ be homogeneous ideals. Then:*

*(a)* $I = (\bigcup_{d=0}^{\infty} I_d)\, R$.

*(b)* $I \subseteq J$ *if and only if* $I_d \subseteq J_d$ *for all* $d$.

*(c)* $I = J$ *if and only if* $I_d = J_d$.

**Proof:** (a) ($\supseteq$) Each $I_d \subseteq I$. Therefore $\bigcup_{d=0}^{\infty} I_d \subseteq I$. This implies that $(\bigcup_{d=0}^{\infty} I_d)\, R \subseteq I$.

($\subseteq$) Let $f \in I$. Then $f = \sum_{d=0}^{e} f_d$ for each $f_d \in R$. Since $I$ is homogeneous, $f_d \in I$ for all $d$. Therefore $f \in I \cap R_d = I_d$. Hence

$$f = \sum_{d=0}^{e} f_d \in \left( \bigcup_{d=0}^{\infty} I_d \right) R.$$

(b) ($\Rightarrow$) If $I \subseteq J$, then $I \cap R_d \subseteq J \cap R_d$, i.e., $I_d \subseteq J_d$.

($\Leftarrow$) If $I_d \subseteq J_d$ for all $d$, then $\bigcup_{d=0}^{\infty} I_d \subseteq \bigcup_{d=0}^{\infty} J_d$. This implies that

$$I = \left( \bigcup_{d=0}^{\infty} I_d \right) R \subseteq \left( \bigcup_{d=0}^{\infty} J_d \right) R = J.$$

(c) This follows from (b) ■

**Lemma 2.3.13** *Let $I \subseteq R$ be a homogeneous ideal. Then* $\operatorname{rad}(I)$ *is a homogeneous ideal.*

**Proof:** Let $f = \sum_{d=0}^{e} f_d \in \operatorname{rad}(I)$ such that each $f_d \in R_d$. We need to show that each $f_d \in \operatorname{rad}(I)$, We will use induction on $e$.

Base Case ($e = 0$): Here $f = f_0$. Since $f \in \operatorname{rad}(I)$, we have $f_0 \in \operatorname{rad}(I)$.

Induction step: Assume for $e \geqslant 1$ the result holds for

$$g = \sum_{d=0}^{e-1} g_d.$$

Since $f \in \operatorname{rad}(I)$, there exists $m \geqslant 1$ such that $f^m \in I$. Now write

$$f^m = f_0^m + \cdots + f_e^m \in I$$
$$f^m = (f^m)_0 + (f^m)_1 + \cdots (f^m)_{me}$$

such that each $(f^m)_j \in R_j$. Then $f_e^m = (f^m)_{me}$. Since $I$ is homogeneous we have $f_e^m \in I$ and so $f_e \in \text{rad}(I)$. Note that $\text{rad}(I)$ is an ideal, therefore $f - f_e \in \text{rad}(I)$ and

$$f - f_e = \sum_{d=0}^{e-1} f_d.$$

By the induction hypothesis $f_i \in \text{rad}(I)$ for $i = 1, \dots, e-1$. ∎

## 2.4 The Zariski Topology on $\mathbb{P}_k^n$

**Definition 2.4.1** *The Zariski topology on* $\mathbb{P}_k^n$

Let $S \subseteq R$ be a set of homogeneous elements of $R$. Define

$$V(S) := \{\underline{a} \in \mathbb{P}_k^n \mid f(\underline{a}) = 0 \text{ for all } f \in S\}.$$

- Closed sets: $V(S)$
- Open sets: $\mathbb{P}_k^n \setminus V(S)$.

Let $f \in R$ be homogeneous: $U_f := \mathbb{P}_k^n \setminus V(f)$ is the "principal open set."

If $T \subseteq R$ (not necessarily homogeneous), then

$$V(T) := V(\{t \in T \mid t \text{ is homogeneous}\}).$$

If $S = \{f_1, \dots, f_m\}$, then $V(S) = V(f_1, \dots, f_m)$.

**Definition 2.4.2** Lines in $\mathbb{P}_k^1$: $V(ax_1 + bx_2 + cx_0)$. Projective ellipse: $V\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2\right)$ or $V\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - x_0^2\right)$. If $x_0 = 1$ (so a copy of $\mathbb{A}_k^2$) we recover our original lines and ellipses. See Example 1.1.6.

**Example 2.4.3** Closed sets in $\mathbb{P}_k^1$ are $V(1) = \emptyset$, $V(0) = \mathbb{P}_k^1$ and (any) finite sets. If $f$ is non-constant, we need to show $V(f)$ is finite. Let $(a_0 : a_1) \in \mathbb{P}_k^1$.

First type of point: $a_0 = 1$. Then $f(\underline{a}) = f(1, a_1)$. So $a_1$ is a root of $f(1, x_1) \neq 0$.

$$\begin{aligned} f &= c_0 x_0^d + c_1 x_0^{d-1} x_1 + \cdots + c_{d-1} x_0 x_1^{d-1} + c_d x_1^d \\ f(1, x_1) &= c_0 + c_1 x_1 + \cdots + x_{d-1} x_1^{d-1} + c_d x_1^d. \end{aligned}$$

Since $f$ is non-constant, $f \neq 0$. Therefore some $c_i \neq 0$. Thus there are only a finite number of roots. Hence $f$ has only a finite number of roots of the form $(1 : a_1)$.

Second type of point: $a_0 = 0$. So assume $a_1 = 1$. Then either $(0 : 1)$ is a root of $f$ or not. Either way you get at most one more point. Therefore there are only finitely many. If $S \neq \emptyset$ is a set of homogeneous polynomials, let $f \in S$ be non-constant. Then $V(S) \subseteq V(f)$ is finite. (Note that Proposition 2.4.14 implies that any finite set is closed.)

**Lemma 2.4.4** *If $T \subseteq T' \subseteq R$, then $V(T) \supseteq V(T')$.*

**Proof:** Exercise. ■

**Lemma 2.4.5** *Let $S \subseteq R$ be the set of homogeneous elements and $I = (S)R$. Then $f$ is homogeneous of degree $d$ and in $I$ if an only if there exists homogeneous polynomials $g_1, \dots, g_m \in R$ and $s_1, \dots, s_m \in S$ such that $f = \sum_i g_i s_i$ and $\deg(g_i) + \deg(s_i) = d$ for all $i$.*

**Proof:** ($\Rightarrow$) Let $f \in I$ be homogeneous of degree $d$. Then we can write $f = g_1 s_1 + \cdots + g_p s_p$ such that each $g_i \in R$ and each $s_i$ is homogeneous of degree $\delta_i$. Hence

$$g_i = g_{i\,0} + \cdots + g_{i\,e}$$

such that $g_{i\,j} \in R_j$. Now we can write $f$ as

$$\begin{aligned} f &= \sum_j g_{i\,j} s_i + \cdots + \sum_j g_{p\,j} s_p \\ &= g_{1\,d-\delta_1} s_1 + g_{2\,d-\delta_2} s_2 + \cdots + g_{p\,p-\delta_p} + (\underbrace{\text{ remaining } g_{i\,j} s_i\text{'s }}_{\text{has no terms of degree } d}) \end{aligned}$$

where $g_{i\,j} s_i$ is homogeneous of degree $j + \delta_i$ and $g_{p\,j} s_p$ is homogeneous of degree $j + \delta_p$ Therefore the last terms must sum to 0 since $f$ only has terms of degree $d$. Therefore

$$f = g_{1\,d-\delta_1} s_1 + g_{2\,d-\delta_2} s_2 + \cdots + g_{p\,p-\delta_p}.$$

($\Leftarrow$) In this case $f$ is homogeneous and in $I$ since all $g_i \cdot s_i$ are homogeneous of degree $d$ and $s_i \in I$. ■

**Lemma 2.4.6** *Let $S \subseteq R$ consist of homogeneous elements of $R$ and let $I = (S)R$. Then $V(I) = V(S)$.*

**Proof:** ($\subseteq$) Note that $I = (S)R \supseteq S$. By Lemma 2.4.4 we have $V(I) \subseteq V(S)$.

($\supseteq$) Let $\underline{a} \in V(S)$ and let $f \in I$ be homogeneous. Then we can write

$$f = g_1 s_1 + \cdots + g_p s_p$$

such that each $g_i \in R$ is homogeneous and $s_i \in S$ by Lemma 2.4.5. Then

$$f(\underline{a}) = g_1(\underline{a}) \underbrace{s_1(\underline{a})}_{=0} + \cdots + g_p(\underline{a}) \underbrace{s_p(\underline{a})}_{=0} = 0.$$

Hence $\underline{a} \in V(I)$. ■

**Lemma 2.4.7** *Let $I_\lambda \subseteq R$ be homogeneous ideals ($\lambda \in \Lambda$). Then the ideals $\bigcap_{\lambda \in \Lambda} I_\lambda$ and $\sum_{\lambda \in \Lambda} I_\lambda$ are homogeneous. If $\Lambda = \{1, \dots, m\}$ then $I_1 \cdots I_m$ is homogeneous.*

**Proof:** $I_\lambda = (S_\lambda)\, R$ where $S_\lambda$ consists of homogeneous polynomials. Then

$$\sum_{\lambda \in \Lambda} I_\lambda = \left( \bigcup_{\lambda \in \Lambda} S_\lambda \right) R$$

where $\bigcup_{\lambda\in\Lambda} S_\lambda$ is a set of homogeneous polynomials. Thus it is a homogeneous ideal.

If $\Lambda = \{1,\dots,m\}$, then $I_1 \cdots I_m = (\{s_1,\dots,s_m \mid s_i \in S_1 \text{ for all } i\})\, R$ is generated by a set of homogeneous polynomial. To show that $\bigcup_{\lambda\in\Lambda} I_\lambda$ is homogeneous, let

$$f = f_0 + f_1 + \cdots + f_d \in \bigcap_{\lambda\in\Lambda} I_\lambda$$

such that $f_i \in R_i$ for all $i$. We need to show that $f_i \in \bigcap_{\lambda\in\Lambda} I_\lambda$ for all $i$. Since $f \in \bigcap_{\lambda\in\Lambda} I_\lambda$, we have $f \in I_\lambda$ for all $\lambda$. Since $I_\lambda$ is homogeneous $f_i \in I_\lambda$ for all $\lambda$. Therefore $f_i \in \bigcap_{\lambda\in\Lambda} I_\lambda$. ■

**Proposition 2.4.8** *Let $S_1,\dots,S_m, S_\lambda \subseteq R$ consisting of homogeneous elements of $R$. $I_j = (S_j)R$, $I_\lambda = (S_\lambda)R$ are homogeneous ideals:*

1. $V(S_1) \cup \cdots \cup V(S_m) = V(I_1) \cup \cdots \cup V(I_m) = V(I_1 \cap \cdots \cap I_m) = V(I_1 \cdots I_m)$.
2. $\bigcap_{\lambda\in\Lambda} V(S_\lambda) = \bigcap_{\lambda\in\Lambda} V(I_\lambda) = V\left(\sum_{\lambda\in\Lambda} I_\lambda\right) = V\left(\bigcup_{\lambda\in\Lambda} S_\lambda\right)$.
3. *The set of projective algebraic subsets is closed under finite unions and arbitrary intersections.*

**Proof:** Exercise ■

**Fact 2.4.9** Let $S \subseteq R$ be a set of homogeneous polynomials in $R$. Then $V(S) = V\left((S)R\right)$

**Definition 2.4.10** If $f \in R$ is homogeneous, then $V(f)$ is a *projective hypersurface.*

**Corollary 2.4.11** *Every algebraic subset of $\mathbb{P}^n_k$ is a finite intersection of projective hypersurfaces.*

**Proof:** $V(S) = V(I) = V(f_1,\dots,f_m)$ where $(S)R = I = (f_1,\dots,f_m)R$ and each $f_i$ is homogeneous. ■

**Proposition 2.4.12** $\{\underline{a}\} = V\left((\{a_i x_j - a_j x_i \mid 0 \leqslant i < j \leqslant n\})R\right) = V\left(\{a_\ell x_j - a_j x_\ell \mid 0 \leqslant j \leqslant n\}\right)$ *for some $a_\ell \neq 0$ where $\underline{a} = (a_0 : a_1 : \cdots : a_n)$.*

**Proof:** We first note that $V\left(\{a_i x_j - a_j x_i\}_{i,j}\right) \subseteq V\left(\{a_\ell x_j - a_j x_\ell\}_j\right)$ since $a_\ell x_j - a_j x_\ell = \pm a_i x_{j'} - a_{j'} x_i$ for some $i, j'$

Next we will show $\{\underline{a}\} \subseteq V\left(\{a_i x_j - a_j x_i\}_{i,j}\right)$. If we evaluate $a_i x_j - a_j x_i$ at $\underline{a}$ we get $a_i a_j - a_j a_i = 0$. Therefore $\{\underline{a}\} \subseteq V\left(\{a_i x_j - a_j x_i\}_{i,j}\right)$.

For the last containment assume that $\underline{b} \in \mathbb{P}^n_k$ satisfies $a_\ell x_j - a_j x_\ell$ for all $\ell$. Then

$$a_\ell b_j - a_j b_\ell = 0$$

for $a_\ell \neq 0$. Therefore $b_j = \frac{a_j b_\ell}{a_\ell} = \frac{b_\ell}{a_\ell} a_j$. Now set $\lambda = \frac{b_\ell}{a_\ell}$. Then

$$\begin{aligned}\underline{a} = (a_0 : a_1 : \cdots : a_n) &= (\lambda a_0 : \lambda a_1 : \cdots : \lambda a_n)\\ &= (b_0 : b_1 : \cdots : b_n) = \underline{b}.\end{aligned}$$

Hence $V\left(\{a_\ell x_j - a_j x_\ell\}_\ell\right) \subseteq \{\underline{a}\}$. ■

**Theorem 2.4.13** *The Zariski topology on $\mathbb{P}^n_k$ is a topology.*

**Proof:** Similar to Theorem 1.2.2. ■

**Proposition 2.4.14** *Every finite subset of $\mathbb{P}^n_k$ is closed.*

**Proof:** Exercise ■

**Example 2.4.15** The Zariski topology on $\mathbb{P}^1_k$ is the cofinite topology. See Example 2.4.3 and Proposition 2.4.14.

**Proposition 2.4.16** *Every open subset in $\mathbb{P}^n_k$ is a finite union of principal open sets.*

**Proof:** Similar to Theorem 1.2.4 ■

**Corollary 2.4.17** *$\mathbb{P}^n_k$ is compact.*

**Proof:** Exercise ■

**Corollary 2.4.18** *$\{$principal open sets of $\mathbb{P}^n_k\}$ is a basis for the Zariski topology.*

**Proof:** Similar to Corollary 1.2.5

**Fact 2.4.19** $U_f \cap U_g = U_{fg}$ for all $f, g \in R$.

**Proof:** Similar to Fact 1.2.6. ■

**Theorem 2.4.20** *If $k$ is infinite and $U, U' \subseteq R$ are non-empty open subsets, then $U \cap U' \neq \emptyset$.*

**Proof:** Similar to Theorem 1.2.8. ■

**Corollary 2.4.21** *If $k$ is infinite, then $\mathbb{P}^n_k$ is <u>not</u> Hausdorff.*

**Proof:** Similar to Corollary 1.2.9. ■

**Fact 2.4.22** If $\underline{a}, \underline{b} \in \mathbb{P}^n_k$ and $\underline{a} \neq \underline{b}$, then there exists a neighborhood $U \subseteq \mathbb{P}^n_k$ of $\underline{a}$ such that $\underline{b} \notin U$.

**Proof:** Set $U = \mathbb{P}^n_k \setminus \{\underline{b}\}$. Then $U$ is the compliment of a closed subset. Therefore $U$ is open with $\underline{a} \in U$ and $\underline{b} \notin U$. ■

<u>Note</u>: $U_{b_i x_j - b_j x_i}$ will work for some choice of $i$ and $j$.

## 2.5 Geometric Ideals in $\mathbb{P}_k^n$

**Definition 2.5.1** Let $V \subseteq \mathbb{P}_k^n$. Then define

$$I(V) := (\{\text{homogeneous } f \in R \mid f(\underline{a}) = 0 \text{ for all } \underline{a} \in V\})\, R.$$

We write $I(\underline{a}_1, \dots, \underline{a}_m)) = I(\{\underline{a}_1, \dots, \underline{a}_m\})$.

Note: $I(V)$ is a homogeneous ideal of $R$ since it is generated by a set of homogeneous polynomials.

**Example 2.5.2** $I(\emptyset) = R$, and $I(\mathbb{P}_k^n) = 0$ if $k$ is infinite.

**Fact 2.5.3** (Division Algorithm for homogeneous polynomials) Let $f$ be a homogeneous polynomial of degree $d$, and let $b$ be a non-zero homogeneous polynomial of degree 1. Write $b = b_0X_0 + \cdots b_nX_n$ such that each $b_i$ is in $k$, and assume that $b_j \neq 0$ for some $j$. Then there exist polynomials $q$ and $r$ such that $f = qb + r$ where $q$ is homogeneous of degree $d-1$, and $r$ is homogeneous of degree $d$ and constant with respect to $X_j$.

**Proof:** Exercise. ■

**Proposition 2.5.4** *Let $\underline{a} \in \mathbb{P}_k^n$. Then $I(\underline{a}) = (\{a_ix_j - a_jx_i \mid 0 \leqslant i < j \leqslant n\})\, R$.*

**Proof:** ($\supseteq$) $\underline{a}$ satisfies each $a_ix_j - a_jx_i$. Therefore each $a_ix_j - a_jx_i \in I(\underline{a})$.

($\subseteq$) First let $J := (\{a_ix_j - a_jx_i \mid 0 \leqslant i < j \leqslant n\})\, R$. Since $I(\underline{a})$ is homogeneous it is generated by the set of its homogeneous elements. We need to show every homogeneous element of $I(\underline{a})$ is in $J$. Let $f \in I(\underline{a})_d$. Then $f(\underline{a}) = 0$. Assume $a_\ell \neq 0$. We will use the division algorithm with

$$a_\ell x_0 - a_0x_\ell, a_\ell x_1 - a_1x_\ell, \dots, a_\ell x_n - a_nx_\ell.$$

Now $f = q_1 \cdot (a_\ell x_0 - a_0x_\ell) + r_1$ such that $q_1, r_1$ are homogeneous and $r_1$ is constant with respect to $x_0$. Then we can write $r_1 = q_2 \cdot (a_\ell x_1 - a_1x_\ell) + r_2$ such that $q_2, r_2$ where $r_2$ is constant with respect to $x_0$ and $x_1$. We repeat this process to get

$$r_i = q_i\,(a_\ell x_i - a_ix_\ell) + r_{i+1}$$

for all $i \neq \ell$. Therefore

$$\begin{aligned} f = {} & q_0\,(a_\ell x_0 - a_0x_\ell) + \cdots + q_{\ell-1}\,(a_\ell x_{\ell-1} - a_{\ell-1}x_\ell) \\ & + q_{\ell+1}\,(a_\ell x_{\ell+1} - a_{\ell+1}x_\ell) + \cdots + q_n\,(a_\ell x_n - a_nx_\ell) + r \end{aligned}$$

where $r$ is constant with respect to all $x_i \neq x_\ell$. Therefore $r = c \cdot x_\ell^d$ and hence $0 = r(\underline{a}) = c \cdot a_\ell^d$ where $a_\ell \neq 0$. Thus $c = 0 \Rightarrow r = 0$. ■

**Proposition 2.5.5** *Let $V, V_1, \dots \subseteq \mathbb{P}_k^n$.*

*(a) If $V \subseteq V_1$, then $I(V) \supseteq I(V_1)$.*

*(b)* $I(V_1 \cup \cdots \cup V_m) = I(V_1) \cap \cdots \cap I(V_m)$.

*(c)* $I\left(\bigcup_{\lambda\in\Lambda} V_\lambda\right) \supseteq \sum_{\lambda\in\Lambda} I(V_\lambda)$.

*(d)* $\operatorname{rad}(I(V)) = I(V)$.

**Proof:**

(a) Assume $V = V_1$. To show $I(V) \supseteq I(V_1)$, it suffices to show $I(V)_d \supseteq I(V_1)_d$ for all $d$. Let $f \in I(V_1)_2$. Then $f(\underline{a}) = 0$ for all $\underline{a} \in V_1$. Therefore $\underline{a} \in V$ since $V \subseteq V_1$. Hence $f \in I(V) \cap R_d = I(V)_d$.

(b) through (d) are proved similarly to Proposition 1.3.3. ■

**Proposition 2.5.6** *Let $I \subseteq R$ be a homogeneous ideal and $V \subseteq \mathbb{P}^n_k$. Then*

*(a)* $I \subseteq \operatorname{rad}(I) \subseteq I(V(I))$.

*(b)* $V \subseteq \overline{V} = V(I(V))$.

*(c)* $V(I(V(I))) = V(I)$.

*(d)* $I(V(I(V))) = I(V)$.

**Proof:** Similar to the proof of Proposition 1.3.4. ■

## 2.6 Projective Nullstellensatz

**Theorem 2.6.1 (Projective Nullstellensatz)** *Let $I \subseteq R = k[x_0, \ldots, x_n]$ be a homogeneous ideal such that $k = \overline{k}$. Then*

*(a) If $V(I) \neq \emptyset$, then $I(V(I)) = \operatorname{rad}(I)$.*

*(b) Let $\mathfrak{M} = (x_0, \ldots, x_n)R$ which is maximal. The following are equivalent:*

*(i)* $V(I) = \emptyset$.

*(ii)* $\operatorname{rad}(I) \supseteq \mathfrak{M}$.

*(iii)* $\operatorname{rad}(I) = \mathfrak{M}$ *or* $\operatorname{rad}(I) = R$.

*(iv)* $\operatorname{rad}(I) = \mathfrak{M}$ *or* $I = R$.

*(v) for all $i = 0, \ldots, n$ there exists $e_i \geqslant 1$ such that $x_i^{e_i} \in I$.*

**Proof:** (a) Define $I_{\text{aff}}$ and $V_{\text{aff}}$ for points in $\mathbb{A}^n_k$ and $I_{\text{proj}}$ and $V_{\text{proj}}$ for points in $\mathbb{P}^n_k$. Assume that $V_{\text{proj}}(I) \neq \emptyset$. By Proposition 2.5.6 $\operatorname{rad}(I) \subseteq I_{\text{proj}}(V_{\text{proj}}(I))$.

<u>Claim</u>: $\operatorname{rad}(I) \subseteq I_{\text{proj}}(V_{\text{proj}}(I)) \subseteq I_{\text{aff}}(V_{\text{aff}}(I)) = \operatorname{rad}(I)$.

To prove the claim let $f \in I_{\text{proj}}(V_{\text{proj}}(I))$.

**Case 1:** Assume $f \in R_d$. We need to show $f \in I_{\text{aff}}(V_{\text{aff}}(I))$. By definition we need to show $f(\underline{a}) = 0$ for all $\underline{a} \in V_{\text{aff}}(I)$. Since $V_{\text{proj}}(I) \neq \emptyset$ we know $d \geqslant 1$ (or else $f = 0$).

**Sub-case 1:** $\underline{a} = \underline{0}$. Since $f$ is homogeneous of degree $d \geqslant 1$ we have $f(\underline{a}) = f(\underline{0}) = 0$.

**Sub-case 2:** $\underline{a} \neq \underline{0}$. Then some $a_j \neq 0$. Therefore $\underline{a}$ represents a point in $\underline{\underline{a}} \in \mathbb{P}_k^n$. Let $\underline{a} = (a_0, a_1, \ldots, a_n)$. Then we write $\underline{\underline{a}} = (a_0 : a_1 : \cdots : a_n)$.

Let $g \in I_e \subseteq I$. Then since $\underline{a} \in V_{\text{aff}}(I)$, we have $g(\underline{\underline{a}}) = g(\underline{a}) = 0$. Therefore $\underline{\underline{a}} \in V_{\text{proj}}(\text{I})$.

Now since $f \in I_{\text{proj}}(V_{\text{proj}}(I))_d$ we have $f(\underline{a}) = f\left(\underline{\underline{a}}\right) = 0$. Hence $f \in I_{\text{aff}}(V_{\text{aff}}(I))$ and hence Case 1 is satisfied.

**Case 2:** (general case) Let $f = \sum_{i=1}^{m} f_i \in I_{\text{proj}}(V_{\text{proj}}(I))$. Since $I_{\text{proj}}(V_{\text{proj}}(I))$ is a homogeneous ideal, each $f_i \in I_{\text{proj}}(V_{\text{proj}}(I))$. Hence case 1 implies that $f_i \in V_{\text{aff}}(I_{\text{aff}}(I))$. Therefore

$$f = \sum_{i=0}^{m} f_i \in I_{\text{aff}}(V_{\text{aff}}(I)).$$

Now the original Nullstellensatz gives us $I_{\text{aff}}(V_{\text{aff}}(I)) = \text{rad}(I)$. Thus the claim is satisfied and we must have that $I_{\text{proj}}(V_{\text{proj}}(I)) = \text{rad}(I)$.

(b) We note that (ii) $\Leftrightarrow$ (iii) is because $\mathfrak{M}$ is maximal, and (iii) $\Leftrightarrow$ (iv) is because $\text{rad}(I) = R$ if and only if $I = R$. For (ii) $\Rightarrow$ (v), if $\mathfrak{M} \subseteq \text{rad}(I)$, then for all $i = 0, \ldots, n$ there exists $e_i \geqslant 1$ such that $x_i^{e_i} \in I$. We will now prove (v) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv).

(v) $\Rightarrow$ (i): Assume (v) and suppose $\underline{a} \in V_{\text{proj}}(I)$ such that $\underline{a} = (a_0 : a_1 : \cdots : a_n)$. Then $\underline{a}$ satisfies every polynomial in $I$, e.g. $x_i^{e_i}$. But $\underline{a} \in \mathbb{P}_k^n$ implies $a_i \neq 0$ for some $i$ contradicting that $\underline{a}$ satisfies $x_i^{e_i}$.

(i) $\Rightarrow$ (iv): Assume $V_{\text{proj}}(I) = \emptyset$. If $I = R$, then we are done. So assume that $I \neq R$. Then $I$ contains no units, i.e. no non-zero constants. Therefore $I_0 = I \cap R_0 = 0$. Hence every polynomial in $I$ has constant term 0. Thus $I \subseteq \mathfrak{M}$. We now have

$$\text{rad}(I) \subseteq \text{rad}(\mathfrak{M}) = \mathfrak{M}.$$

($\subseteq$) Note that by the original Nullestensatz $I_{\text{aff}}(V_{\text{aff}}(I)) = \text{rad}(I) \subseteq \mathfrak{M} = \mathfrak{m}_{\underline{0}}$. Therefore $\underline{0} \in V_{\text{aff}}(I)$.

<u>Claim</u>: $V_{\text{aff}}(I) = \{\underline{0}\}$.

Note that for this claim we have already shown ($\supseteq$). For the other containment let $\underline{a} \in V_{\text{aff}}(I)$ and suppose $\underline{a} \neq 0$. Then $\underline{a}$ represents some point $\underline{\underline{a}} \in \mathbb{P}_k^n$, say $\underline{\underline{a}} = (a_0 : a_1 : \cdots : a_n)$. Then $\underline{a}$ satisfies all $f \in I$ and hence $\underline{a}$ satisfies all homogeneous $f \in I$. Therefore $\underline{\underline{a}}$ satisfies all homogeneous $f \in I$. Thus $\underline{\underline{a}} \in V_{\text{proj}}(I)$ contradicting that $V_{\text{proj}}(I) = \emptyset$. Hence $V_{\text{aff}}(I) = \{\underline{0}\}$.

Now by the original Nullestensatz we have $\text{rad}(I) = I_{\text{aff}}(V_{\text{aff}}(I)) = I_{\text{aff}}(\{\underline{0}\}) = \mathfrak{m}_{\underline{0}} = \mathfrak{M}$. ■

**Definition 2.6.2** The ideal $\mathfrak{M}$, as in Theorem 2.6.1, is the *irrelevant maximal ideal* of $R$.

**Fact 2.6.3** $\mathfrak{M}$ is the unique homogeneous maximal ideal of $R$ and it contains every homogeneous ideal $I \neq R$.

**Corollary 2.6.4** *Assume* $k = \overline{k}$. *Then*

(a) $\{$*non-empty closed subsets of* $\mathbb{P}_k^n\} \rightleftarrows \{$*homogeneous ideals* $I \subseteq R$ *such that* $I = \text{rad}(I) \subsetneq \mathfrak{M}\}$ *defined by*

$$\begin{aligned} V &\longmapsto I(V) \\ V(I) &\longleftarrow\!\shortmid I. \end{aligned}$$

*(b) If $I = \mathrm{rad}(I) \subsetneq \mathfrak{M}$ is homogeneous, then $V(I) \neq \emptyset$.*

**Proof:** Similar to Corollary 1.4.2. ■

## 2.7 Irreducible Closed Subsets

**Lemma 2.7.1** *Let $I \subsetneq R$ be a homogeneous ideal. The following are equivalent:*

*(a) $I$ is prime.*

*(b) For all homogeneous ideals $J, K$: if $JK \subseteq I$, then $J \subseteq I$ or $K \subseteq I$.*

*(c) For all homogeneous $f, g \in R$: if $f, g \in I$, then $f \in I$ or $g \in I$.*

**Proof:** (a) $\Rightarrow$ (c) is by definition of a prime ideal.

(c) $\Rightarrow$ (b): Assume (c) and suppose $J \not\subseteq I$ and $K \not\subseteq I$ where $J$ and $K$ are homogeneous ideals. Then there exists $d, e \geqslant 0$ such that $J_d \not\subseteq I_d$ and $K_e \not\subseteq I_e$. Therefore there exists homogeneous $f \in J \setminus I$ and $g \in K \setminus I$. (c) then implies $f, g \notin I$. Therefore $fg \in JK$ and hence $JK \not\subseteq I$.

(b) $\Rightarrow$ (a): Assume (b). Let $f, g \in R \setminus I$. We need to show that $fg \notin I$. Now we will write

$$\begin{aligned} f &= f_0 + f_1 + \cdots + f_d \\ g &= g_0 + g_1 + \cdots + g_e. \end{aligned}$$

Therefore some $f_i \notin I$ and $g_j \notin I$.

**Case 1:** $f_d, g_e \notin I$. Note that if we take $J = (f_d)R$ and $K = (g_e)R$, then (b) implies that we have $f_d g_e \notin I$. Now

$$fg = f_0 g_0 + (f_1 g_0 + f_0 g_1) + \cdots + f_d g_e.$$

If $fg \in I$, then since $I$ is homogeneous we have $f_d g_e \in I$ which is a contradiction.

**Case 2:** (general case) Assume without loss of generality that $f_\alpha \in I$ for all $\alpha > i$ and $g_\beta \in I$ for all $\beta > j$. Then

$$\begin{aligned} f &= f_0 + f_1 + \cdots + f_i + \underbrace{f_{i+1} + \cdots + f_d}_{:=\tilde{f} \in I} \\ g &= g_0 + g_1 + \cdots + g_j + \underbrace{g_{j+1} + \cdots + g_e}_{:=\tilde{g} \in I}. \end{aligned}$$

Now define $f' := f - \tilde{f} = f_0 + \cdots f_i \notin I$ and $g' := g - \tilde{g} = g_0 + \cdots + g_j \notin I$. Case 1 now implies that $f'g' \notin I$. Therefore

$$\begin{aligned} fg &= (f' + \tilde{f})(g' + \tilde{g}) \\ &= \underbrace{f'g'}_{\notin I} + \underbrace{f'\tilde{g}}_{\in I} + \underbrace{\tilde{f}g'}_{\in I} + \underbrace{\tilde{f}\tilde{g}}_{\in I} \notin I. \end{aligned}$$

Therefore $fg \notin I$. ■

**Theorem 2.7.2** *Let $V \subseteq \mathbb{P}_k^n$ be closed. Then $V$ is irreducible if and only if $I(V)$ is prime.*

**Proof:** ($\Rightarrow$) Assume that $V$ is irreducible. Then since $V \neq \emptyset$ we have that $I(V) \neq R$. To show $I(V)$ is prime, we will use Lemma 2.7.1. Let $f, g \in R$ be homogeneous such that $fg \in I(V)$. Let $J = I(V)+fR$ and $K = I(V)+gR$. These are homogeneous ideals because $I(V)$ is homogeneous and $f, g$ are homogeneous. As before: $JK \subseteq I(V) \subseteq J \cap K$. This implies that

$$V(JK) \supseteq \underbrace{V(I(V))}_{=V} \supseteq \underbrace{V(J \cap K)}_{=V(JK)}.$$

Therefore $V = V(JK) = V(J) \cup V(K)$. Since $V$ is irreducible either $V = V(J)$ or $V = V(K)$. This now implies that either $I(V) = I(V(J)) \ni f$ or $I(V) = I(V(K)) \ni g$. Hence $I(V)$ is prime.

($\Leftarrow$) Similar to the proof of Theorem 1.5.5. ■

**Corollary 2.7.3** *If $R = k[x_0, \ldots, x_n]$ such that $k = \overline{k}$ and $\mathfrak{p} \subset R$, $\mathfrak{p} \neq \mathfrak{M} = (x_0, \ldots, x_n)R$, is prime, then $V(\mathfrak{p})$ is irreducible.*

**Proof:** Theorem 2.6.1 implies that $V(\mathfrak{p}) \neq \emptyset$ and $I(V(\mathfrak{p})) = \mathrm{rad}(\mathfrak{p}) = \mathfrak{p}$. Therefore Theorem 2.7.2 implies that $V(\mathfrak{p})$ is irreducible. ■

**Proposition 2.7.4** *If $k$ is finite, then the only irreducible closed subsets of $\mathbb{P}_k^n$ are $\{\underline{\underline{a}}\}$.*

**Proof:** Similar to Proposition 1.5.9. ■

**Corollary 2.7.5** *If $k$ is finite, then $\mathbb{P}_k^n$ is reducible.*

**Lemma 2.7.6** *Let $k$ be infinite and $f, g \in R$ be homogeneous. If $f(\underline{\underline{a}})g(\underline{\underline{a}}) = 0$ for all $\underline{\underline{a}} \in \mathbb{P}_k^n$, then either $f = 0$ or $g = 0$.*

**Proof:** Let $0 \neq f, g \in R$ be homogeneous. It suffices to show that $f(\underline{a})g(\underline{a}) = 0$ for all $\underline{a} \in \mathbb{A}_k^{n+1}$. Then Lemma 1.2.7 implies that either $f = 0$ or $g = 0$, a contradiction.

First assume that $f$ is constant, say $f = c$, then $f$ is a unit. Therefore $g(\underline{\underline{a}}) = 0$ for all $\underline{\underline{a}} \in \mathbb{P}_k^n$. If $g$ is constant, then we are done. Suppose that $g$ is not constant. Since $g$ is homogeneous, it follows that $g(\underline{0}) = 0$. If $\underline{0} \neq \underline{a} \in \mathbb{A}_k^{n+1}$, then $\underline{a}$ represents $\underline{\underline{a}} \in \mathbb{P}_k^n$, so $g(\underline{a}) = g(\underline{\underline{a}}) = 0$. Thus $g(\underline{a}) = 0$ for all $\underline{a} \in \mathbb{A}_k^n$. Therefore $g = 0$ since $k$ is infinite. Similarly if $g$ is constant, then $f = 0$.

Now assume $f, g$ are both are non-constant. Let $\underline{a} \in \mathbb{A}_k^{n+1}$. If $\underline{a} = \underline{0}$, then $f(\underline{a})g(\underline{a}) = 0$. So assume that $\underline{a} \neq 0$. Then $\underline{a}$ represents $\underline{\underline{a}} \in \mathbb{P}_k^n$ and

$$f(\underline{a})g(\underline{a}) = f(\underline{\underline{a}})g(\underline{\underline{a}}) = 0.$$

Therefore either $f(\underline{a}) = f(\underline{\underline{a}}) = 0$ or $g(\underline{a}) = g(\underline{\underline{a}}) = 0$. ■

**Proposition 2.7.7** *If $k$ is infinite, then $\mathbb{P}_k^n$ is irreducible.*

**Proof:** Suppose $\mathbb{P}_k^n = V(I) \cup V(J) = V(IJ)$ where $I, J \subseteq R$ are homogeneous ideals. If $I = 0$, then $V(I) = V(0) = \mathbb{P}_k^n$. Similarly if $J = 0$.

Now assume that $I, J \neq 0$. Then there exists homogeneous $0 \neq f \in I$ and $0 \neq g \in J$. Then

$$\mathbb{P}_k^n = V(I) \cup V(J) \subseteq V(f) \cup V(g) \subseteq \mathbb{P}_k^n.$$

Hence $f, g \neq 0$ are homogeneous such that $f(\underline{a})g(\underline{a}) = 0$ for all $\underline{a} \in \mathbb{P}_k^n$. Therefore by Lemma 2.7.6 $f = 0$ or $g = 0$ which contradicts that both $f, g \neq 0$. ∎

**Theorem 2.7.8** *$\mathbb{P}_k^n$ is noetherian.*

**Proof:** Similar to Theorem 1.5.15. ∎

**Corollary 2.7.9** *Every closed subset $V \subseteq \mathbb{P}_k^n$ is a union of a finite number of irreducible closed subsets. Also if you assume*

$$V = V_1 \cup V_2 \cup \cdots \cup V_m$$

*such that $V_i \nsubseteq V_j$ (for all $i \neq j$), then the decomposition is unique up to the order of the $V_i$'s.*

**Proof:** Exercise ∎

**Theorem 2.7.10** *Let $V \subseteq \mathbb{P}_k^n$ be closed. Then* $\dim(V) \leqslant \dim(R/I(V)) - 1$.

**Proof:** Let $\emptyset \neq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m \subseteq V$ be a chain of irreducible subsets in $V$. Then

$$I(V) \subseteq \underbrace{I(V_m) \subseteq \cdots \subseteq I(V_1) \subseteq I(V_0) \subsetneq \mathfrak{M}}_{\text{prime}}.$$

We note that $I(V_0) \subsetneq \mathfrak{M}$ since $V_0 \neq \emptyset$. Also note that if $V_i$ is irreducible, then $I(V_i)$ is prime by Theorem 2.7.2 and $V(I(V_i)) = V_i$. Therefore $I(V_i) \subsetneq I(V_{i-1})$ for all $i \geqslant 1$. Hence $R/I(V)$ has a chain of primes with $m+1$ links. Thus $\dim(R/I(V)) \geqslant m + 1$. ∎

**Fact 2.7.11** If $k$ is infinite, then $\dim(\mathbb{P}_k^n) = n$.

**Proof:** Exercise ∎

**Definition 2.7.12** Define $k[V] := R/I(V)$. This is the *homogeneous coordinate ring* for $V$.

## 2.8 Regular Functions

**Definition 2.8.1** Let $f : \mathbb{P}_k^n \to \mathbb{P}_k^m$ be a function. Then $f$ is a *regular* function if there exists homogeneous polynomials

$$f_0, f_1, \ldots, f_m \in k[x_0, x_1, \ldots, x_n]$$

(all have the same degree) such that $f(\underline{a}) = (f_0(\underline{a}) : f_1(\underline{a}) : \cdots : f_m(\underline{a}))$ for all $\underline{a} \in \mathbb{P}_k^n$.

**Fact 2.8.2** The function defined in Definition 2.8.1 is a well-defined function as long as

$$V(f_0) \cap \cdots \cap V(f_m) = \emptyset.$$

We need this to make sure that $f(\underline{a}) \neq (0 : 0 : \cdots : 0)$.

Since $f_i$ is homogeneous of degree $d$, we have $f_i(\lambda a_0, \lambda a_1, \ldots, \lambda a_n) = \lambda^d f_i(a_0, a_1, \ldots, a_n)$. Let

$$\underline{b} = (\lambda a_0 : \lambda a_2, : \cdots : \lambda a_n) = \underline{a}$$

(assuming $\lambda \neq 0$ in $k$). Then

$$\begin{aligned} f(\underline{b}) &= (f_0(\underline{b}) : f_1(\underline{b}) : \cdots : f_n(\underline{b})) \\ &= \left(\lambda^d f_0(\underline{a}) : \lambda^d f_1(\underline{a}) : \cdots : \lambda^d f_n(\underline{a})\right) \\ &= f(\underline{a}). \end{aligned}$$

**Fact 2.8.3** If $f$ is a regular function, then $f$ is continuous. Also the composition of regular functions is regular.

**Examples of Continuous Functions**

1. $\mathbb{A}^n_k \xrightarrow{F} \mathbb{A}^m_k$ such that $F(\underline{a}) = (f_1(\underline{a}), \ldots, f_n(\underline{a}))$ where each $f_i$ is a polynomial.
2. $\mathbb{P}^n_k \xrightarrow{F} \mathbb{P}^m_k$ such that $F(\underline{a}) = (f_0(\underline{a}) : \cdots : f_m(\underline{a}))$ and such that each $f_i \in k[X_1, \ldots, X_m]_d$ and $V(f_0) \cap \cdots \cap V(f_m) = \emptyset$.
3. Let $U = U_{f_0} \cup U_{f_1} \cup \cdots \cup U_{f_m} \subseteq \mathbb{P}^n_k$ where $f_i \in k[X_0, \ldots, X_n]_d$. Define $U \xrightarrow{F} \mathbb{P}^m_k$ where $F(\underline{a}) := (f_0(\underline{a}) : \cdots : f_m(\underline{a}))$.
4. $f_i : \mathbb{A}^n_k \to \mathbb{P}^n_k$, where $f_i(\underline{a}) = (a_1 : \cdots a_{i-1} : 1 : a_{i+1} : \cdots : a_n)$.
5. $g_i : \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k$ where $g_i(\underline{a}) = (a_0 : \cdots : a_{i-1} : 0 : a_{i+1} : \cdots a_{n-1})$.

## 2.9 Finding Irreducible Components

**Definition 2.9.1** Let $R = k[x_0, \ldots, x_n]$, and $I \subsetneq R$ be a homogeneous ideal. $I$ is *h-irreducible* if it is irreducible with respect to homogeneous ideals, i.e., for all homogeneous ideals $J, K$ if $I = J \cap K$, then $I = J$ or $I = K$.

**Fact 2.9.2** If $I \subsetneq R$ is homogeneous and irreducible, then $I$ is h-irreducible.

**Proposition 2.9.3** *If $I \subsetneq R$ is h-irreducible and $a, b \in R$ are homogeneous such that $ab \in I$, then $a \in I$ or $b^m \in I$ for some $m$.*

**Proof:** Note that $(I : b) \subseteq (I : b^2) \subseteq \cdots$. Since $R$ is noetherian, the ascending chaing condition implies that for some $m$ we have $(I : b^m) = (I : b^{m+1})$. Then as in the proof of Proposition 1.6.4 we have

$$I = \underbrace{(I + aR)}_{\text{homogeneous}} \cap \underbrace{(I + b^m R)}_{\text{homogeneous}}$$

Therefore since $I$ is h-irreducible, we have $I = I + aR \ni a$ or $I = I + b^m R \ni b^m$ ■

**Lemma 2.9.4** *If $I \subseteq R$ is a homogeneous ideal, then $I$ is prime if and only if $I$ is h-irreducible and $I = \operatorname{rad}(I)$.*

**Proof:** ($\Rightarrow$) Assume that $I$ is prime. Then by Proposition 1.6.5, we have that $I$ is irreducible and $I = \operatorname{rad}(I)$. Hence by Fact 2.9.2, $I$ is h-irreducible.

($\Leftarrow$) Now assume that $I$ is h-irreducible and $I = \operatorname{rad}(I)$. Then $I \subsetneq R$. To show $I$ is prime, let $a, b \in R$ be homogeneous such that $ab \in I$. Proposition 2.9.3 implies that $a \in I$ or $b^m \in I$ for some $m$. Since $I = \operatorname{rad}(I)$, we must have $a \in I$ or $b \in I$, so $I$ is prime. ■

**Lemma 2.9.5** *Let $I \subsetneq R$ be an h-irreducible ideal. Then $\operatorname{rad}(I)$ is a prime and homogeneous ideal.*

**Sketch of Proof:** Assume that $I$ is an h-irreducible ideal. Then it is homogemeneous. Hence by Lemma 2.3.13 we have $\operatorname{rad}(I)$ is homogeneous. Also, $\operatorname{rad}(I)$ is prime by checking homogeneous elements; see the proof of Proposition 1.6.6. ■

**Definition 2.9.6** Let $I \subsetneq R$ be a homogeneous ideal. An *h-irreducible decomposition* $I = Q_1 \cap \cdots \cap Q_m$ such that each $Q_i$ is h-irreducible. The decomposition is *irredundant* if for all $i \neq j$, then $Q_i \not\subseteq Q_j$.

**Proposition 2.9.7** *Let $I \subsetneq R$ be a homogeneous ideal. Then $I$ has an irredundant h-irreducible decomposition.*

**Proof:** Similar to the proof of Proposition 1.6.9 ■

**Proposition 2.9.8** *If $I \subsetneq R$ is a homogeneous ideal and $P_1, \ldots, P_m, Q_1, \ldots, Q_\ell$ are homogeneous primes such that*

$$P_1 \cap \cdots \cap P_m = Q_1 \cap \cdots \cap Q_\ell$$

*and if $i \neq j$, then $P_i \not\subseteq P_j$ and $Q_i \not\subseteq Q_j$. Then $m = \ell$ and there exists $\sigma \in S_m$ such that $P_i = Q_{\sigma(i)}$ for all $i$.*

**Proof:** This is a corollary to Theorem 1.5.20. ■

**Proposition 2.9.9** *Let $I \subsetneq R$ be a homogeneous ideal. Then $I = \operatorname{rad}(I)$ if and only if for some prime ideals $P_1, \ldots, P_m$ such that $I = P_1 \cap \cdots \cap P_m$.*

**Proof:** Similar to the proof of Proposition 1.6.11 (a). ■

**Corollary 2.9.10** *If $I = \operatorname{rad}(I)$ is homogeneous and $\mathfrak{p}_1 \ldots, \mathfrak{p}_m$ are prime ideals such that $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ is an irredundant decomposition, then each $\mathfrak{p}_i$ is homogeneous.*

**Proof:** Let $P_1, \dots, P_\ell$ be homogeneous prime ideals such that $I = P_1 \cap \cdots \cap P_\ell$ is an irredundant decomposition. The uniqueness implies that $\mathfrak{p}_i = P_{\sigma(i)}$ is homogeneous. ■

**Question 2.9.11** How do we find irreducible components for $V(I) \subseteq \mathbb{P}^n_k$?

**Theorem 2.9.12** *Assume that $k = \overline{k}$ and $I \subsetneq R$ is a homogeneous ideal such that $V(I) \neq \emptyset$ and $\operatorname{rad}(I) = P_1 \cap \cdots \cap P_m$ is an irredundant prime decomposition (hence $P_i$'s are homogeneous). Then the irreducible components of $V(I)$ are $V(P_1), \dots, V(P_m)$.*

**Proof:** Note that each $V(P_i) \neq \emptyset$. So we need to show $P_i \neq \mathfrak{M}$. If $P_i = \mathfrak{M}$, then $P_j \subseteq \mathfrak{M} = P_i$ for all $j$. Since the decomposition is irredundant we have that $I = P_i = \mathfrak{M}$. But this contradicts that $V(I) \neq \emptyset$. ■

**Theorem 2.9.13** *Assume $k = \overline{k}$ and $I \subseteq R$ is homogeneous such that $V(I) \neq \emptyset$. Let $I = Q_1 \cap \cdots \cap Q_m$ be an h-irreducible decomposition. Then*

$$\operatorname{rad}(I) = \operatorname{rad}(Q_1) \cap \cdots \cap \operatorname{rad}(Q_m).$$

*Remove redundancies and reorder to assume $\operatorname{rad}(I) = \operatorname{rad}(Q_1) \cap \cdots \cap \operatorname{rad}(Q_\ell)$ is an irredundant homogeneous prime decomposition. Then the irreducible components of $V(I)$ are $V(\operatorname{rad}(Q_1)) = V(Q_1), \dots, V(\operatorname{rad}(Q_\ell) = V(Q_\ell)$.*

**Theorem 2.9.14** *Let $I \subsetneq R$ be a homogeneos ideal such that $V(I) \neq \emptyset$. Then $I(V(I)) = \operatorname{rad}(I(V(I))) \subsetneq R$, so $I(V(I)) = P_1 \cap \cdots \cap P_m$ such that each $P_i$ is homogeneous and prime. Assume the decomposition is irredundant. Then the irreducible components of $V(I)$ are $V(P_1), \dots, V(P_m)$.*

**Proof:** Similar to the proof of Proposition 1.6.18 ■

CHAPTER 3

# Sheaves

## 3.1 Presheaves

**Definition 3.1.1** Let $X$ be a topological space. A *presheaf of abelian groups* on $X$ is a rule $\mathcal{G}$:

1. For all open subsets $U \subseteq X$, $\mathcal{G}(U)$ is an abelian group.

2. For every pair of open subsets $U \subseteq U' \subseteq X$ there is an abelian group homomorphism $\mathcal{G} : (U', U) : \mathcal{G}(U') \to \mathcal{G}(U)$.

such that

(a) $\mathcal{G}(\emptyset) = 0$;

(b) $\mathcal{G}(U, U) = \mathrm{id}_{\mathcal{G}(U)}$;

(c) If $U \subseteq U' \subseteq U'' \subseteq X$ are open, then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{G}(U'') & \xrightarrow{\mathcal{G}(U'', U')} & \mathcal{G}(U') \\ & \searrow^{\mathcal{G}(U'', U)} & \downarrow{\scriptstyle \mathcal{G}(U', U)} \\ & & \mathcal{G}(U). \end{array}$$

The group $\mathcal{G}(U)$ is the group of "sections" of $\mathcal{G}$ over $U$. The homomorphisms $\mathcal{G}(U', U)$ is the "restriction map." To rephrase: Let $X$ be a topological space. Set $\mathrm{Open}(X)$ = the category of open subsets of $X$ with morphisms = containments. Set $\mathcal{A}b$ = the category of abelian groups with group homomorphisms = $\mathbb{Z}$-Mod. A presheaf of abelian groups is nothing more than a contravariant functor $\mathcal{G} : \mathrm{Open}(X) \to \mathcal{A}b$.

**Example 3.1.2** For all open sets $U \subseteq X$, let

$$\mathcal{G}(U) = \{\text{continuous functions } U \to \mathbb{R}\}.$$

If $U \subseteq U'$, then $\mathcal{G}(U', U) =$ restriction of functions $\mathcal{G}(U') \to \mathcal{G}(U)$ where $f \mapsto f|_U$. Define addition in $\mathcal{G}(U)$ pointwise:

$$(f+g)(u) = f(u) + g(u).$$

This makes $\mathcal{G}$ into a presheaf of abelian groups on $X$.

**Definition 3.1.3** A *presheaf of rings* on $X$ is a presheaf $\mathcal{G}$ of additive abelian groups such that each $\mathcal{G}(U)$ is a ring and each $\mathcal{G}(U', U)$ is a ring homomorophism (i.e. $\mathcal{G} : \mathrm{Open}(X) \to \mathrm{Rings}$).

Similarly for a *presheaf of $k$-algebras*:

- A *$k$-algebra* is a ring homomorphism $\phi : k \to R$.
- A *morphism of $k$-algebras* is a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} k & \xrightarrow{\phi} & R \\ & \searrow^{\psi} & \downarrow f \\ & & S. \end{array}$$

**Example 3.1.4** Let $k \to k[x_1, \ldots, x_n]$ be a $k$-algebra and let $J \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then the following diagram commutes:

$$\begin{array}{ccc} k & \longrightarrow & k[x_1, \ldots, x_n] \\ & \searrow & \downarrow \\ & & k[x_1, \ldots, x_n]/J. \end{array}$$

**Example 3.1.5** The presheaf $\mathcal{G}$ from Example 3.1.2 is a presheaf of $\mathbb{R}$-algebras. Multiplication in $\mathcal{G}(U)$ is pointwise $(f \cdot g)(u) = f(u) \cdot g(u)$. The map $\mathbb{R} \xrightarrow{\psi_U} \mathcal{G}(U)$ where $r \mapsto$ constant function, i.e., $\psi_U(r) : U \to \mathbb{R}$ is given by $\psi_U(r)(u) = r$. The map $\mathcal{G}(U', U) : \mathcal{G}(U') \to \mathcal{G}(U)$ by restriction is a $k$-algebra homomorphism (constant$|_U$ is constant on $U$).

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\psi_{U'}} & \mathcal{G}(U') \\ & \searrow^{\psi_U} & \downarrow \mathcal{G}(U', U) \\ & & \mathcal{G}(U). \end{array}$$

## 3.2 Regular Functions on $\mathbb{A}^n_k$

**Definition 3.2.1** Let $U \subseteq \mathbb{A}^n_k$ be open and let $\phi : U \to k$ be a function.

1. $\phi$ is *regular* at $\alpha \in U$ if there exists an open set $U(\alpha) \subseteq U$ such that $\alpha \in U(\alpha)$ and there exists polynomials $p_\alpha, q_\alpha \in R = k[x_1, \ldots, x_n]$ such that $U(\alpha) \subseteq U_{q_\alpha}$ and for all $\beta \in U(\alpha)$ we have $\phi(\beta) = \frac{p_\alpha(\beta)}{q_\alpha(\beta)}$.

   In other words, there exists $\frac{p_\alpha}{q_\alpha} \in k(x_1, \ldots, x_n)$ such that $\phi = \frac{p_\alpha}{q_\alpha}$ on some neighborhood of $\alpha$ on which $\frac{p_\alpha}{q_\alpha}$ is a well-defined function.

2. $\phi$ is *regular* on $U$ if it is regular at $\alpha$ for all $\alpha \in U$.

**Example 3.2.2** Let $f \in R = k[x_1, \ldots, x_n]$ and $\emptyset \neq U \subseteq \mathbb{A}^n_k$ be an open set. Then $\frac{f}{1} = f$ defines a regular function on $U$.

**Example 3.2.3** Let $f \in R$ and consider $U_f \subseteq \mathbb{A}^n_k$. Then for all $g \in R$ and for all $m \in \mathbb{N}$, we have $\frac{g}{f^m}$ is regular on $U_f$.

**Proposition 3.2.4** *Let $\lambda \in k$. Then the constant function $U \to k$, given by $u \mapsto \lambda$, is regular.*

**Proof:** Let $\lambda \in k \subseteq R$ be a polynomial. Then Example 3.2.2 implies that the constant function is regular. ∎

**Lemma 3.2.5** *Let $\emptyset \neq U \subseteq \mathbb{A}^n_k$ be open and let $\phi : U \to k$ be a function. Then $\phi$ is regular on $U$ if and only if there exists an open cover $U_1 \cup \cdots \cup U_m = U$ and $p_1, \ldots, p_m, q_1, \ldots, q_m \in R$ such that $q_i \neq 0$ for all $i$ and $U_i \subseteq U_{q_i}$ for all $i$ and $\phi|_{U_i} = \left.\frac{p_i}{q_i}\right|_{U_i}$ for all $i$.*

**Proof:** ($\Leftarrow$) Assume that such $p_1, q_i$, and $U_i$ exist. Then for every $\alpha \in U$ there exists $i$ such that $\alpha \in U_{q_i}$ since $U = U_1 \cup \cdots \cup U_m$. Set

$$\begin{aligned} U(\alpha) &= U_i \\ p_\alpha &= p_1 \\ q_\alpha &= q_i. \end{aligned}$$

Then $\phi|_{U(\alpha)} = \phi|_{U_i} = \left.\frac{p_i}{q_i}\right|_{U_i} = \left.\frac{p_\alpha}{q_\alpha}\right|_{U(\alpha)}$. Hence $\phi$ is regular on $U$.

($\Rightarrow$) Assume that $\phi$ is regular. We use the notation from Definition 3.2.1. Then $U = \bigcup_{\alpha \in U} U(\alpha)$ since $\alpha \in U(\alpha) \subseteq U$ for all $\alpha \in U$. Since $\mathbb{A}^n_k$ is noetherian, we have $U$ is noetherian and hence this open cover has a finite sub-cover

$$U = U(\alpha_1) \cup \cdots \cup U(\alpha_m).$$

Set $q_i = q_{\alpha_i}$, $p_i = p_{\alpha_i}$, and $U_i = U(\alpha_i)$. Then $U = \bigcup_{i=1}^m U(\alpha_i) = \bigcup_{i=1}^m U_i$, and

$$\phi|_{U(\alpha_i)} = \left.\frac{p_i}{q_i}\right|_{U(\alpha_i)} = \left.\frac{p_i}{q_i}\right|_{U_i}.$$

∎

**Exercise 3.2.6** Let $X$ be a topological space and $Z \subseteq X$ be a subset and let $X = U_1 \cup \cdots \cup U_m$ be an open cover. Then $Z$ is closed in $X$ if and only if $Z \cap U_i$ is closed in $U_i$ for all $i$.

**Proposition 3.2.7** *Let $\phi : U \to k = \mathbb{A}^1_k$ be a regular function. Then $\phi$ is continuous.*

**Proof:** We need to show $\phi^{-1}(V)$ is closed where $V$ is closed. So let $V \subseteq k$ be a nontrivial closed subset. Then $V$ is finite by Example 1.1.7, say $V = \{a_1, \dots, a_m\}$. Hence

$$\phi^{-1}(\{a_1, \dots, a_n\}) = \phi^{-1}(a_1) \cup \cdots \cup \phi^{-1}(a_m).$$

So it suffices to show that $\phi^{-1}(a)$ is closed in $U$ for all $a \in k$. Since $\phi$ is regular, Lemma 3.2.5 provides an open cover $U = U_1 \cup \cdots \cup U_m$ and nonzero $q_1, \dots, q_m \in R$ such that each $U_i \subseteq U_{q_i}$ and there exists $p_1, \dots, p_m \in R$ such that $\phi|_{U_i} = \frac{p_i}{q_i}\Big|_{U_i}$ for all $i$. Then for all $\alpha \in U_i$, we have

$$\begin{aligned} \alpha \in \phi^{-1}(a) &\Leftrightarrow \phi(\alpha) = a \\ &\Leftrightarrow \frac{p_i(\alpha)}{q_i(\alpha)} = a \\ &\Leftrightarrow p_i(\alpha) = a \cdot q_i(\alpha) \\ &\Leftrightarrow \alpha \in \underbrace{V(p_i - aq_i)}_{\text{closed in } \mathbb{A}^n_k}. \end{aligned}$$

Hence $\phi^{-1}(a) \cap U_i = V(p_i - aq_i) \cap U_i$ is closed in $U_i$. Therefore by Exercise 3.2.6 we have that $\phi^{-1}(a) \subseteq U$ is closed. ■

**Proposition 3.2.8** *If $\phi : U \to k$ is regular and $U' \subseteq U$ is open, then $\phi|_{U'} : U' \to k$ is regular.*

**Proof:** Let $U = U_1 \cup \cdots \cup U_m$ be an open cover such that $\phi|_{U_i} = \frac{p_i}{q_i}$ for all $i$. Set $U_i' = U_i \cap U'$. Then

$$(\phi|_{U'})|_{U_i'} = \phi|_{U_i'} = (\phi_{U_i})|_{U_i'} = \left( \frac{p_i}{q_i}\Big|_{U_i} \right)\Bigg|_{U_i'} = \frac{p_i}{q_i}\Big|_{U_i'}.$$

Therefore $\phi|_{U_i}$ is regular. ■

**Definition 3.2.9** Let $U \subseteq \mathbb{A}^n_k$ be open. Define $\mathcal{O}_{\mathbb{A}^n_k}(U) = \{\text{regular functions } U \to k\}$ and define $\mathcal{O}_{\mathbb{A}^n_k}(\emptyset) = 0$. Let $U' \subseteq U$ and define $\mathcal{O}_{\mathbb{A}^n_k}(U, U') : \mathcal{O}_{\mathbb{A}^n_k}(U) \to \mathcal{O}_{\mathbb{A}^n_k}(U')$ by $\phi \mapsto \phi|_{U'}$. Note that Proposition 3.2.8 implies that this map is well-defined.

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