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REFLEXIVITY AND CONNECTEDNESS

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Abstract. Given a finitely generated module over a commutative noetherian ring that satisfies certain reflexivity conditions, we show how failure of the semidualizing property for the module manifests in a disconnection of the prime spectrum of the ring.

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1. Introduction. Throughout this paper, R is a non-zero commutative noetherian ring with identity and all R-modules are unital.

An R-module A is a *semidualizing* if the natural homothety map $\chi_A^R\colon R \to \operatorname{Hom}_R(A, A)$ is an isomorphism and $\operatorname{Ext}_R^i(A, A) = 0$ for all $i \geqslant 1$. These gadgets, and their cousins the semidualizing complexes, are useful for studying dualities. For instance, their applications include Grothendieck's local duality [**14**, **15**], progress by Avramov-Foxby [**4**] and Sather-Wagstaff [**16**] on the composition question for local ring homomorphisms of finite G-dimension, and progress by Sather-Wagstaff [**17**] on Huneke's question on the behaviour of Bass numbers of local rings.

The starting point for the current paper is the following straightforward result, wherein a finitely generated R-module M is *totally A-reflexive* if the natural biduality map $\delta_A^M\colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, A), A)$ is an isomorphism and $\operatorname{Ext}_R^i(\operatorname{Hom}_R(M, A), A) = 0 = \operatorname{Ext}_R^i(M, A)$ for all $i \geqslant 1$:

FACT 1.1. *For a finitely generated R-module A, the next conditions are equivalent:*

(i) *A is a* semidualizing *R-module,*
(ii) *R is a* totally A-reflexive *R-module and*
(iii) *A is totally A-reflexive and* $\operatorname{Ann}_R(A) = 0$.

For perspective, we sketch the proof of the implication (iii) $\Longrightarrow$ (i). When A is totally A-reflexive, we have $\operatorname{Ext}_R^i(A, A) = 0$ for all $i \geqslant 1$. Thus it remains to assume that $\operatorname{Ann}_R(A) = 0$ and show that the homothety map $\chi_A^R\colon R \to \operatorname{Hom}_R(A, A)$ is an isomorphism. Since $\operatorname{Ker}(\chi_A^R) = \operatorname{Ann}_R(A) = 0$, it remains to show that $\operatorname{Coker}(\chi_A^R) = 0$. This is equivalent to showing that $\operatorname{Ext}_R^i(\operatorname{Coker}(\chi_A^R), A) = 0$ for all $i \geqslant 0$, as we have $\operatorname{Supp}_R(A) = V(\operatorname{Ann}_R(A)) = V(0) = \operatorname{Spec}(R)$. Consider the short exact sequence $0 \to R \xrightarrow{\chi_A^R} \operatorname{Hom}_R(A, A) \to \operatorname{Coker}(\chi_A^R) \to 0$. The induced long exact sequence in $\operatorname{Ext}_R^i(-, A)$, along with the vanishing $\operatorname{Ext}_R^i(\operatorname{Hom}_R(A, A), A) = 0 = \operatorname{Ext}_R^i(R, A)$ for $i \geqslant 1$, shows that $\operatorname{Ext}_R^i(\operatorname{Coker}(\chi_A^R), A) = 0$ for all $i \geqslant 2$. Furthermore, this provides another exact sequence

$$0 \to \operatorname{Coker}(\chi_A^R)^* \to A^{**} \xrightarrow{(\chi_A^R)^*} A \to \operatorname{Ext}_R^1(\operatorname{Coker}(\chi_A^R), A) \to 0$$

where $(-)^* = \operatorname{Hom}_R(-, A)$. The biduality map $\delta_A^A \colon A \to \operatorname{Hom}_R(\operatorname{Hom}_R(A, A), A)$ is an isomorphism by assumption. Furthermore, the composition $(\chi_A^R)^* \circ \delta_A^A$ is the identity id_A, and it follows that $(\chi_A^R)^*$ is an isomorphism. Thus, the displayed sequence gives the vanishing $\operatorname{Ext}_R^i(\operatorname{Coker}(\chi_A^R), A) = 0$ for the remaining values $i = 0, 1$, completing the proof.

It is straightforward to show that the annihilator condition in item (iii) of Fact 1.1 is necessary: if A is totally A-reflexive, then A need not be semidualizing. For instance, if $A = 0$, then A is totally A-reflexive but is not semidualizing. A slightly less trivial example is the following:

Example 1.2. Let R_1, R_2 be non-zero commutative noetherian rings with identity, and set $R = R_1 \times R_2$. Then the R-module $A = R_1 \times 0$ is totally A-reflexive but is not semidualizing. Moreover, given any semidualizing R_1-module A_1, the R-module $A = A_1 \times 0$ is totally A-reflexive but is not semidualizing.

The point of this paper is to show that this is the only way for this to occur. Specifically, we prove the following in 3.8:

Theorem 1.3. *Let A be a non-zero finitely generated R-module that is totally A-reflexive and not semidualizing. Then there are commutative noetherian rings $R_1, R_2 \neq 0$ with identity such that $R \cong R_1 \times R_2$, and there is a semidualizing R_1-module A_1 such that $A \cong A_1 \times 0$. In particular,* $\operatorname{Spec}(R)$ *is disconnected.*

If R is local or a domain, then $\operatorname{Spec}(R)$ is connected. Hence, if A is non-zero and totally A-reflexive, then A must be semidualizing. The local version of this is actually a key point of the proof of Theorem 1.3, which is contained in Theorem 3.1 below. The version for domains is documented in Corollary 3.10. Note that our results also give other conditions on A that imply that $\operatorname{Spec}(R)$ is disconnected or that A is semidualizing. These conditions are akin to those studied in [**5**, **12**].

2. Background.

We begin this section with some background information.

Definition 2.1. We work in the derived category $\mathcal{D}(R)$ where each R-complex X is indexed homologically: $X = \cdots \to X_i \xrightarrow{\partial_i^X} X_{i-1} \to \cdots$. An R-complex X is *homologically bounded* if $\operatorname{H}_i(X) = 0$ for all but finitely many i. The complex X is *homologically finite* if it is homologically bounded and $\operatorname{H}_i(X)$ is finitely generated for all i. The ith *suspension* of X is $\Sigma^i X$. Isomorphisms in $\mathcal{D}(R)$ are identified with the symbol $\simeq$. Two R-complexes X and Y are *shift-isomorphic*, written $X \sim Y$, if there is an integer i such that $X \simeq \Sigma^i Y$. The *large support* of X is $\operatorname{Supp}_R(X) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid X_\mathfrak{p} \not\simeq 0\}$. Given two R-complexes X and Y, the *right derived Hom complex* and *left derived tensor product complex* of X and Y are denoted $\mathbf{R}\operatorname{Hom}_R(X, Y)$ and $X \otimes_R^{\mathbf{L}} Y$, and $\operatorname{Ext}_R^i(X, Y) := \operatorname{H}^i(\mathbf{R}\operatorname{Hom}_R(X, Y))$.

If $(R, \mathfrak{m}, k)$ is local, then the *Bass series* and *Poincaré series* of a homologically finite R-complex X are the formal Laurent series

$$I_R^X(t) = \sum_{i\in\mathbb{Z}} \operatorname{rank}_k(\mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(k, X)))t^i$$
$$P_X^R(t) = \sum_{i\in\mathbb{Z}} \operatorname{rank}_k(\mathrm{H}_i(k \otimes_R^{\mathbf{L}} X))t^i.$$

The following complexes and the classes that they define originate in work of Auslander-Bridger [**2**, **3**], Avramov-Foxby [**4**], Christensen [**6**], Enochs-Jenda-Xu [**7**], Foxby [**8**, **9**], Golod [**13**], Vasconcelos [**18**], and Yassemi [**19**].

DEFINITION 2.2. Let A, N be R-complexes.

(a) A is *semidualizing* if it is homologically finite and the natural homothety morphism $\chi_A^R \colon R \to \mathbf{R}\mathrm{Hom}_R(A, A)$ is an isomorphism in $\mathcal{D}(R)$.
(b) A is *tilting* if it is semidualizing and has finite projective dimension.
(c) N is *derived A-reflexive* if N and $\mathbf{R}\mathrm{Hom}_R(N, A)$ are homologically finite and the natural biduality morphism $\delta_N^A \colon N \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(N, A), A)$ is an isomorphism in $\mathcal{D}(R)$.
(d) N is in the *Bass class* $\mathcal{B}_A(R)$ if N and $\mathbf{R}\mathrm{Hom}_R(A, N)$ are homologically bounded and the natural evaluation morphism $\xi_N^A \colon A \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(A, N) \to N$ is an isomorphism in $\mathcal{D}(R)$.
(e) N is in the *Auslander class* $\mathcal{A}_A(R)$ if N and $A \otimes_R^{\mathbf{L}} N$ are homologically bounded and the natural morphism $\gamma_N^A \colon N \to \mathbf{R}\mathrm{Hom}_R(A, A \otimes_R^{\mathbf{L}} N)$ is an isomorphism in $\mathcal{D}(R)$.

The following facts are straightforward to verify.

FACT 2.3. *Let A be a finitely generated R-module, and let N be an R-module.*

(a) *A is semidualizing as an R-complex if and only if it is semidualizing as an R-module.*
(b) *A is tilting as an R-complex if and only if it is a rank-1 projective R-module.*
(c) *An R-module that is totally A-reflexive is derived A-reflexive. If N has a finite resolution by totally A-reflexive R-modules, then it is derived A-reflexive; the converse holds when A is semidualizing [**19**].*
(d) *If the natural map $A \otimes_R \mathrm{Hom}_R(A, N) \to N$ is bijective and $\mathrm{Ext}_R^i(A, N) = 0 = \mathrm{Tor}_i^R(A, \mathrm{Hom}_R(A, N))$ for all $i \geqslant 1$, then $N \in \mathcal{B}_A(R)$; the converse holds when A is semidualizing by* [**6**, (4.10) Observation].
(e) *If the natural map $N \to \mathrm{Hom}_R(A, A \otimes_R N)$ is bijective and $\mathrm{Tor}_i^R(A, N) = 0 = \mathrm{Ext}_R^i(A, A \otimes_R N)$ for all $i \geqslant 1$, then $N \in \mathcal{A}_A(R)$; the converse holds when A is semidualizing by* [**6**, (4.10) Observation].

LEMMA 2.4. *Assume that R is local, and let A and B be homologically finite R-complexes such that $A \not\simeq 0$. Then the following conditions are equivalent:*

(i) *$B \simeq 0$,*
(ii) *$A \otimes_R^{\mathbf{L}} B \simeq 0$,*
(iii) *$\mathbf{R}\mathrm{Hom}_R(A, B) \simeq 0$, and*
(iv) *$\mathbf{R}\mathrm{Hom}_R(B, A) \simeq 0$.*

Proof. For $n =$ii,iii,iv, the implications (i) $\Longrightarrow$ (n) are standard. For the converses, we suppose that $B \not\simeq 0$, and conclude that $A \otimes_R^{\mathbf{L}} B \not\simeq 0$, $\mathbf{R}\mathrm{Hom}_R(A, B) \not\simeq 0$, and

$\mathbf{R}\mathrm{Hom}_R(B, A) \not\simeq 0$. For instance, this follows from the Bass series and Poincaré series computations in [**4**, Lemma (1.5.3)]. □

LEMMA 2.5. *Assume that R is local, and let A, X and Y be homologically finite R-complexes such that $A \not\simeq 0$. Given a morphism $f: X \to Y$ the following conditions are equivalent:*

(i) *f is an isomorphism in $D(R)$,*
(ii) *$A \otimes^{\mathbf{L}}_R f$ is an isomorphism in $D(R)$,*
(iii) *$\mathbf{R}\mathrm{Hom}_R(A, f)$ is an isomorphism in $D(R)$, and*
(iv) *$\mathbf{R}\mathrm{Hom}_R(f, A)$ is an isomorphism in $D(R)$.*

Proof. Apply Lemma 2.4 to the mapping cone $B := \mathrm{Cone}(f)$. □

FACT 2.6. *Let A a homologically finite R-complex. Then the following conditions are equivalent:*

(i) *A is semidualizing over R,*
(ii) *There is an isomorphism $\mathbf{R}\mathrm{Hom}_R(A, A) \simeq R$ in $D(R)$,*
(iii) *For each maximal ideal $\mathfrak{m} \subset R$, there is an isomorphism $\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(A_\mathfrak{m}, A_\mathfrak{m}) \simeq R_\mathfrak{m}$ in $D(R_\mathfrak{m})$,*
(iv) *R is derived A-reflexive,*
(v) *A is derived A-reflexive and $\mathrm{Supp}_R(A) = \mathrm{Spec}(R)$ and*
(vi) *$U^{-1}A$ is semidualizing over $U^{-1}R$ for each multiplicatively closed $U \subseteq R$.*

Indeed, in addition to [**5**, Proposition 3.1]*, it suffices to note that the implications* (2.6) $\Longrightarrow$ (2.6) $\Longrightarrow$ (2.6) *are straightforward.*

REMARK 2.7. Assume that R_1 and R_2 are commutative noetherian rings such that $R \cong R_1 \times R_2$. Using the natural idempotents in R, one checks readily that every R-complex is isomorphic to one of the form $X_1 \times X_2$ where X_i is an R_i-complex for $i = 1, 2$.

For $i = 1, 2$, let X_i, Y_i and Z_i be R_i-complexes. Recall that there are natural isomorphisms in $\mathcal{D}(R)$:

$$\mathbf{R}\mathrm{Hom}_R(X_1 \times X_2, Y_1 \times Y_2) \simeq \mathbf{R}\mathrm{Hom}_{R_1}(X_1, Y_1) \times \mathbf{R}\mathrm{Hom}_{R_2}(X_2, Y_2)$$
$$(X_1 \times X_2) \otimes^{\mathbf{L}}_R (Y_1 \times Y_2) \simeq (X_1 \otimes^{\mathbf{L}}_{R_1} Y_1) \times (X_2 \otimes^{\mathbf{L}}_{R_2} Y_2).$$

From this, it follows that

(a) $X_1 \times X_2$ is semidualizing for R if and only if each X_i is semidualizing for R_i.
(b) $\mathbf{R}\mathrm{Hom}_R(X_1 \times X_2, Y_1 \times Y_2)$ is semidualizing for R if and only if $\mathbf{R}\mathrm{Hom}_R(X_i, Y_i)$ is semidualizing for R_i for $i = 1, 2$.
(c) $X_1 \times X_2$ is derived $Y_1 \times Y_2$-reflexive if and only if each X_i is derived Y_i-reflexive.
(d) $\mathbf{R}\mathrm{Hom}_R(X_1 \times X_2, Y_1 \times Y_2)$ is derived $Z_1 \times Z_2$-reflexive if and only if the complex $\mathbf{R}\mathrm{Hom}_R(X_i, Y_i)$ is derived Z_i-reflexive for $i = 1, 2$.
(e) $X_1 \times X_2 \in \mathcal{B}_{Y_1 \times Y_2}(R)$ if and only if $X_i \in \mathcal{B}_{Y_i}(R_i)$ for $i = 1, 2$.
(f) $X_1 \times X_2 \in \mathcal{A}_{Y_1 \times Y_2}(R)$ if and only if $X_i \in \mathcal{A}_{Y_i}(R_i)$ for $i = 1, 2$.

DEFINITION 2.8. The *semidualizing locus* of a homologically finite R-complex A is

$$\mathrm{SD}_R(A) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid A_\mathfrak{p} \text{ is semidualizing for } R_\mathfrak{p}\}.$$

REMARK 2.9. Let A be a homologically finite R-complex. Then we have

$$\operatorname{Spec}(R) \smallsetminus \operatorname{Supp}_R(\operatorname{Cone}(\chi^R_A)) = \operatorname{SD}_R(A) \subseteq \operatorname{Supp}_R(A).$$

Also, A is semidualizing for R if and only if $\operatorname{SD}_R(A) = \operatorname{Spec}(R)$; see Fact 2.6.

LEMMA 2.10. *Let A be a homologically finite R-complex such that $\mathbf{R}\operatorname{Hom}_R(A, A)$ is homologically finite, that is such that $\operatorname{Ext}^i_R(A, A) = 0$ for $i \gg 0$. Then $\operatorname{SD}_R(A)$ is Zariski open in $\operatorname{Spec}(R)$.*

Proof. As $\mathbf{R}\operatorname{Hom}_R(A, A)$ is homologically finite, so is the mapping cone $\operatorname{Cone}(\chi^R_A)$. So, the set $\operatorname{SD}_R(A) = \operatorname{Spec}(R) \smallsetminus \operatorname{Supp}_R(\operatorname{Cone}(\chi^R_C))$ is open in $\operatorname{Spec}(R)$. □

3. Results. We begin this section with the local version of our main results.

THEOREM 3.1. *Assume that R is local, and let A be a homologically finite R-complex. Then the following conditions are equivalent:*

(i) *A is semidualizing for R,*
(ii) *$\mathbf{R}\operatorname{Hom}_R(A, A)$ is semidualizing for R,*
(iii) *A is derived A-reflexive and $A \not\simeq 0$,*
(iv) *$\mathbf{R}\operatorname{Hom}_R(A, A)$ is derived A-reflexive and $A \not\simeq 0$,*
(v) *$A \in \mathcal{B}_A(R)$ and $A \not\simeq 0$ and*
(vi) *$R \in \mathcal{A}_A(R)$.*

Proof. Note that if A is semidualizing for R, then $A \not\simeq 0$ since $0 \simeq \mathbf{R}\operatorname{Hom}_R(0, 0) \not\simeq R$. Similarly, if $\mathbf{R}\operatorname{Hom}_R(A, A)$ is semidualizing for R, then $A \not\simeq 0$. Thus, for n =ii,iii,iv,v, the implications (i) $\Longrightarrow$ (n) are from [**12**, Theorem 1.3].

(ii) $\Longrightarrow$ (i) Assume that $\mathbf{R}\operatorname{Hom}_R(A, A)$ is semidualizing for R, and consider the following commutative diagram in $\mathcal{D}(R)$.

$$\begin{array}{ccc} R & \xrightarrow[\simeq]{\chi^R_{\mathbf{R}\operatorname{Hom}_R(A,A)}} & \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(A, A), \mathbf{R}\operatorname{Hom}_R(A, A)) \\ {\scriptstyle \chi^R_A}\downarrow & & \downarrow{\scriptstyle \simeq} \\ \mathbf{R}\operatorname{Hom}_R(A, A) & \xrightarrow{\mathbf{R}\operatorname{Hom}_R(\xi^A_A, A)} & \mathbf{R}\operatorname{Hom}_R(A \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(A, A), A) \end{array}$$

The unspecified isomorphism is Hom-tensor adjointness. From this, it follows that there is a monomorphism $R \hookrightarrow \mathrm{H}_0(\mathbf{R}\operatorname{Hom}_R(A, A))$, so $\mathrm{H}_0(\mathbf{R}\operatorname{Hom}_R(A, A)) \neq 0$. From this, we conclude that a minimal free resolution $F \simeq \mathbf{R}\operatorname{Hom}_R(A, A)$ has $F_0 \neq 0$. Thus, there is a coefficient-wise inequality $P^R_{\mathbf{R}\operatorname{Hom}_R(A,A)}(t) \succeq 1$.

From the above diagram, it follows that the composition $\mathbf{R}\operatorname{Hom}_R(\xi^A_A, A) \circ \chi^R_A$ is an isomorphism, hence, so is the induced morphism

$$\mathbf{R}\operatorname{Hom}_R(k, \mathbf{R}\operatorname{Hom}_R(\xi^A_A, A) \circ \chi^R_A) = \mathbf{R}\operatorname{Hom}_R(k, \mathbf{R}\operatorname{Hom}_R(\xi^A_A, A)) \circ \mathbf{R}\operatorname{Hom}_R(k, \chi^R_A)$$

where k is the residue field of R. In particular, the induced map on homology

$$\operatorname{Ext}^i_R(k, R) \to \operatorname{Ext}^i_R(k, \mathbf{R}\operatorname{Hom}_R(A, A))$$

is a monomorphism for each i. This explains the first coefficient-wise inequality in the next sequence:

$$I_R^{\mathbf{R}\mathrm{Hom}_R(A,A)}(t) \succeq I_R^R(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(A,A)}(t) I_R^{\mathbf{R}\mathrm{Hom}_R(A,A)}(t) \succeq I_R^{\mathbf{R}\mathrm{Hom}_R(A,A)}(t).$$

The equality follows from the fact that $\mathbf{R}\mathrm{Hom}_R(A, A)$ is semidualizing, by [**10**, 1.5]. The second coefficient-wise inequality is from the condition $P^R_{\mathbf{R}\mathrm{Hom}_R(A,A)}(t) \succeq 1$ established above. It follows that we have a coefficient-wise equality

$$P^R_{\mathbf{R}\mathrm{Hom}_R(A,A)}(t) I_R^{\mathbf{R}\mathrm{Hom}_R(A,A)}(t) = I_R^{\mathbf{R}\mathrm{Hom}_R(A,A)}(t).$$

From this, we conclude that $P^R_{\mathbf{R}\mathrm{Hom}_R(A,A)}(t) = 1$, so $\mathbf{R}\mathrm{Hom}_R(A, A) \simeq R$ and A is semidualizing by Fact 2.6.

(iii) $\Longrightarrow$ (i) Assume that A is derived A-reflexive and $A \not\simeq 0$. It follows that $\mathbf{R}\mathrm{Hom}_R(A, A)$ is homologically finite. Consider the following commutative diagram in $\mathcal{D}(R)$ where the unspecified isomorphism is Hom-cancellation.

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{\delta^A_A} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(A, A), A) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\chi^R_A, A)} \\ A & \xleftarrow[\simeq]{} & \mathbf{R}\mathrm{Hom}_R(R, A) \end{array}$$

It follows that $\mathbf{R}\mathrm{Hom}_R(\chi^R_A, A)$ is an isomorphism in $\mathcal{D}(R)$, so Lemma 2.5 implies that χ^R_A is an isomorphism in $\mathcal{D}(R)$, thus A is semidualizing.

(iv) $\Longrightarrow$ (iii) Assume that $\mathbf{R}\mathrm{Hom}_R(A, A)$ is derived A-reflexive and $A \not\simeq 0$. In particular, the biduality morphism

$$\delta^A_{\mathbf{R}\mathrm{Hom}_R(A,A)} \colon \mathbf{R}\mathrm{Hom}_R(A, A) \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(A, A), A), A)$$

is an isomorphism in $\mathcal{D}(R)$. From [**5**, 2.2] we conclude that $\mathbf{R}\mathrm{Hom}_R(\delta^A_A, A)$ is an isomorphism in $\mathcal{D}(R)$. Since A and $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(A, A), A)$ are both homologically finite by assumption, Lemma 2.5 implies that δ^A_A is an isomorphism in $\mathcal{D}(R)$. It follows that A is derived A-reflexive.

(v) $\Longrightarrow$ (i) Assume that $A \in \mathcal{B}_A(R)$, and consider the commutative diagram

$$\begin{array}{ccc} A \otimes^{\mathbf{L}}_R R & \xrightarrow{A \otimes^{\mathbf{L}}_R \chi^A_A} & A \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(A, A) \\ {\scriptstyle \simeq}\downarrow & \swarrow{\scriptstyle \xi^A_A} & \\ A & & \end{array}$$

As in the previous paragraphs, Lemma 2.5 implies that A is semidualizing.

(i) $\Longleftrightarrow$ (vi) This follows readily from the next commutative diagram in $\mathcal{D}(R)$.

$$\begin{array}{ccc} R & \xrightarrow{\xi^A_R} & \mathbf{R}\mathrm{Hom}_R(A, A \otimes^{\mathbf{L}}_R R) \\ {\scriptstyle \chi^R_A}\downarrow & \swarrow{\scriptstyle \simeq} & \\ \mathbf{R}\mathrm{Hom}_R(A, A) & & \end{array}$$

See also [**6**, (4.4) Proposition] for one implication. □

REMARK 3.2. In Theorem 3.1 the implications (vi) $\implies$ (i) $\implies$ (n) for n =ii,iii,iv,v,vi do not use the local assumption. The point of much of the remainder of this paper is that the implications (n) $\implies$ (i) fail in general for n =ii,iii,iv,v. Moreover, we explicitly characterize the failure of these implications.

The proof of the next result is similar to the proof of [**11**, Theorem 3.2].

THEOREM 3.3. *Assume that R is local, and let A be a homologically finite R-complex. Then $0 \not\simeq A \in \mathcal{A}_A(R)$ if and only if $A \sim R$.*

Proof. One implication is straightforward: if $A \sim R$, then $\mathcal{A}_A(R)$ contains all homologically bounded complexes, so $A \in \mathcal{A}_A(R)$ and $0 \not\simeq R \sim A$.

For the converse, assume that $0 \not\simeq A \in \mathcal{A}_A(R)$. Shift A if necessary to assume that $\inf\{n \in \mathbb{Z} \mid \mathrm{H}_n(A) \neq 0\} = 0$. Let $P \simeq A$ be a minimal free resolution of A. It follows that $P_i = 0$ for all $i < 0$ and $P_0 \neq 0$. The condition $P \simeq A \in \mathcal{A}_A(R)$ implies that the natural map $\gamma_P^P : P \to \mathrm{Hom}_R(P, P \otimes_R P)$ is a quasiisomorphism, hence it induces the quasiisomorphism in the top row of the next commutative diagram of chain maps where the unspecified isomorphism is Hom-tensor adjointness.

$$\begin{array}{ccc}
\mathrm{Hom}_R(P,P) & \xrightarrow[\simeq]{\mathrm{Hom}_R(P,\gamma_P^P)} & \mathrm{Hom}_R(P, \mathrm{Hom}_R(P, P \otimes_R P)) \\
& \searrow^{\Delta} & \downarrow \cong \\
& & \mathrm{Hom}_R(P \otimes_R P, P \otimes_R P)
\end{array}$$

In degree 0, the composition Δ is given by $f \mapsto P \otimes_R f$. The diagram shows that Δ is a quasiisomorphism.

Given two R-complexes X and Y, let $\Theta_{X,Y} : X \otimes_R Y \to Y \otimes_R X$ by the natural commutativity isomorphism $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$. This is a chain map, hence the fact that Δ is a quasiisomorphism implies that there is a chain map $f : P \to P$ such that $P \otimes_R f : P \otimes_R P \to P \otimes_R P$ is homotopic to $\Theta_{P,P}$.

Let k be the residue field of R, and set $\overline{(-)} = k \otimes_R -$. The previous paragraph implies that $\overline{P \otimes_R f} : \overline{P \otimes_R P} \to \overline{P \otimes_R P}$ is homotopic to $\overline{\Theta_{P,P}}$. Using the natural isomorphism $\overline{- \otimes_R -} \cong \overline{-} \otimes_k \overline{-}$, it follows that $\overline{P} \otimes_k \overline{f} : \overline{P} \otimes_k \overline{P} \to \overline{P} \otimes_k \overline{P}$ is homotopic to $\Theta_{\overline{P},\overline{P}}$. Since P is minimal, the differentials on $\overline{P}$ and $\overline{P} \otimes_k \overline{P}$ are 0, and it follows that $\overline{P} \otimes_k \overline{f} = \Theta_{\overline{P},\overline{P}} : \overline{P} \otimes_k \overline{P} \to \overline{P} \otimes_k \overline{P}$.

We first show that $\overline{P_0} \cong k$. Since $\overline{P_0}$ is a non-zero k-vector space, it suffices to show that $\mathrm{rank}_k(\overline{P_0}) \leqslant 1$. Suppose that $r = \mathrm{rank}_k(\overline{P_0}) \geqslant 2$, and let $x_1, \ldots, x_r \in \overline{P_0}$ be a basis. It follows that $\overline{P_0} \otimes_k \overline{P_0}$ has rank r^2 with basis $\{x_i \otimes x_j \mid i, j = 1, \ldots, r\}$. The equality $\overline{P} \otimes_k \overline{f} = \Theta_{\overline{P},\overline{P}}$ implies that

$$x_2 \otimes x_1 = x_1 \otimes f(x_2) \in \mathrm{Span}_k\{x_1 \otimes x_1, \ldots, x_1 \otimes x_r\}$$

contradicting the linear independence of the given basis for $\overline{P_0} \otimes_k \overline{P_0}$.

We now show that $\overline{P_i} = 0$ for all $i \neq 0$. (It then follows that $A \simeq P \cong R$, as desired.) Let $i \geqslant 1$ and $y \in \overline{P_i}$. With x_1 as in the previous paragraph, the equality $\overline{P} \otimes_k \overline{f} = \Theta_{\overline{P},\overline{P}}$ implies that

$$0 = y \otimes x_1 - x_1 \otimes f(y) \in (\overline{P_i} \otimes_k \overline{P_0}) \oplus (\overline{P_0} \otimes_k \overline{P_i}).$$

Since $i \neq 0$, we have $(\overline{P_0} \otimes_k \overline{P_i}) \cap (\overline{P_i} \otimes_k \overline{P_0}) = 0$, so we conclude that $y \otimes x_1 = 0$ in $\overline{P_i} \otimes_k \overline{P_0}$. Since $0 \neq x_1$ in the vector space $\overline{P_0}$, it follows that $y = 0$. The element $y \in \overline{P_i}$ was chosen arbitrarily, so we conclude that $\overline{P_i} = 0$, as desired. □

Next, we present our non-local results.

COROLLARY 3.4. *Let A be a homologically finite R-complex. Then the following conditions are equivalent:*

(i) *A is semidualizing for R,*
(ii) $\mathbf{R}\mathrm{Hom}_R(A, A)$ *is semidualizing for R,*
(iii) *A is derived A-reflexive and* $\mathrm{Supp}_R(A) = \mathrm{Spec}(R)$,
(iv) $\mathbf{R}\mathrm{Hom}_R(A, A)$ *is derived A-reflexive and* $\mathrm{Supp}_R(A) = \mathrm{Spec}(R)$,
(v) $A \in \mathcal{B}_A(R)$ *and* $\mathrm{Supp}_R(A) = \mathrm{Spec}(R)$ *and*
(vi) $R \in \mathcal{A}_A(R)$.

Proof. Note that conditions (i), (ii) and (vi) all imply that $\mathrm{Supp}_R(A) = \mathrm{Spec}(R)$ since A is homologically finite. The implications (vi) $\iff$ (i) $\implies$ (n) for $n =$ii,iii,iv,v follow from Remark 3.2. For the implications (n) $\implies$ (i) with $n =$ii,iii,iv,v, note that condition (n) localizes; since the semidualizing property is local by Fact 2.6, the desired conclusion follows from Theorem 3.1. □

The next result is proved like the previous one, via Theorem 3.3.

COROLLARY 3.5. *Let A be a homologically finite R-complex. Then $A \in \mathcal{A}_A(R)$ and* $\mathrm{Supp}_R(A) = \mathrm{Spec}(R)$ *if and only if A is a tilting R-complex.*

As we show in 3.8 below, the next result is the key to proving Theorem 1.3.

THEOREM 3.6. *Let A be a homologically finite R-complex. Then the following conditions are equivalent:*

(i) *There are non-zero commutative noetherian rings R_1, R_2 with identity such that $R \cong R_1 \times R_2$, and there is a semidualizing R_1-complex A_1 such that $A \cong A_1 \times 0$,*
(ii) *A is derived A-reflexive and not semidualizing such that $A \not\simeq 0$,*
(iii) $\mathbf{R}\mathrm{Hom}_R(A, A)$ *is derived A-reflexive, A is not semidualizing, and $A \not\simeq 0$ and*
(iv) *$0 \not\simeq A \in \mathcal{B}_A(R)$ and A is not semidualizing.*

In particular, when the above conditions are satisfied, $\mathrm{Spec}(R)$ *is disconnected.*

Proof. (i) $\implies$ (ii) Assume that there are non-zero commutative noetherian rings R_1, R_2 with identity such that $R \cong R_1 \times R_2$, and that there is a semidualizing R_1-complex A_1 such that $A \cong A_1 \times 0$. Since $R_2 \neq 0$, we conclude that 0 is not semidualizing for R_2, so Remark 2.7(a) implies that A is not semidualizing for R. Since A_1 is semidualizing for $R_1 \neq 0$, we conclude that $A \not\simeq 0$, and that A is derived A-reflexive by Remarks 2.7(c) and 3.2.

(ii) $\implies$ (i) Assume that A is derived A-reflexive and not semidualizing such that $A \not\simeq 0$. In particular, the complex $\mathbf{R}\mathrm{Hom}_R(A, A)$ is homologically finite. Lemma 2.10 implies that $\mathrm{SD}_R(A)$ is an open subset of $\mathrm{Spec}(R)$.

We claim that $\mathrm{SD}_R(A) = \mathrm{Supp}_R(A)$. One containment is from Remark 2.9. For the reverse containment, let $\mathfrak{p} \in \mathrm{Supp}_R(A)$. It follows that $A_\mathfrak{p} \not\simeq 0$ is totally $A_\mathfrak{p}$-reflexive, so Theorem 3.1 implies that $A_\mathfrak{p}$ is semidualizing for $R_\mathfrak{p}$, that is $\mathfrak{p} \in \mathrm{SD}_R(A)$.

It follows that $\mathrm{SD}_R(A) = \mathrm{Supp}_R(A)$ is both open and closed in $\mathrm{Spec}(R)$. Since A is not semidualizing, Remark 2.9 shows that $\mathrm{SD}_R(A) = \mathrm{Supp}_R(A) \neq \mathrm{Spec}(R)$. On the other hand, since $A \not\simeq 0$, we have $\mathrm{SD}_R(A) = \mathrm{Supp}_R(A) \neq \emptyset$. It follows that $\mathrm{Spec}(R) = \mathrm{Supp}_R(A) \uplus (\mathrm{Spec}(R) \smallsetminus \mathrm{Supp}_R(A))$ is a disconnection of $\mathrm{Spec}(R)$. A standard result (see, e.g. [**1**, Exercise 1.22]) implies that there are commutative rings R_1 and R_2 such that

(1) $R \cong R_1 \times R_2$ and
(2) Under the natural bijection $\mathrm{Spec}(R) \cong \mathrm{Spec}(R_1) \uplus \mathrm{Spec}(R_2)$, the set $\mathrm{Supp}_R(A)$ corresponds to $\mathrm{Spec}(R_1)$, and $\mathrm{Spec}(R) \smallsetminus \mathrm{Supp}_R(A)$ corresponds to $\mathrm{Spec}(R_2)$.

Remark 2.7 implies that for $i = 1, 2$ there is an R_i-complex A_i such that $A \simeq A_1 \times A_2$. Under the natural bijection $\mathrm{Spec}(R) \cong \mathrm{Spec}(R_1) \uplus \mathrm{Spec}(R_2)$, for each $P \in \mathrm{Spec}(R)$ and its corresponding prime $\mathfrak{p}_i \in \mathrm{Spec}(R_i)$, we have $A_P \simeq (A_i)_{\mathfrak{p}_i}$. Using condition (2) above, it follows that

(3) for each $\mathfrak{p}_1 \in \mathrm{Spec}(R_1)$, corresponding to $P \in \mathrm{Supp}_R(A) = \mathrm{SD}_R(A)$, since A_P is semidualizing for R_P, the complex $(A_1)_{\mathfrak{p}_1}$ is semidualizing for $(R_1)_{\mathfrak{p}_1}$ and
(4) for each $\mathfrak{p}_2 \in \mathrm{Spec}(R_2)$ corresponding to $P \in \mathrm{Spec}(R) \smallsetminus \mathrm{Supp}_R(A)$, we have $(A_2)_{\mathfrak{p}_2} \simeq A_P \simeq 0$.

Because of condition (3), Fact 2.6 implies that A_1 is semidualizing for R_1. And condition (4) implies that $\mathrm{Supp}_{R_2}(A_2) = \emptyset$, so $A_2 \simeq 0$, as desired.

For n =iii,iv, the equivalence (i) $\iff$ (n) is proved similarly. □

THEOREM 3.7. *Let A be a homologically finite R-complex. Then the following conditions are equivalent:*

(i) *$0 \not\simeq A \in \mathcal{A}_A(R)$ and A is not semidualizing for R and*
(ii) *there are non-zero commutative noetherian rings R_1, R_2 with identity such that $R \cong R_1 \times R_2$, and there is a tilting R_1-complex A_1 such that $A \cong A_1 \times 0$.*

Proof. From [**12**, Proposition 4.4], we know that A is tilting if and only if $A_\mathfrak{m} \sim R_\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset R$. Thus, the desired implications follow from Theorem 3.3 as in the proof of Theorem 3.6. □

3.8 *Proof of Theorem 1.3*. Let A be a non-zero totally A-reflexive R-module that is not semidualizing. Then A is derived A-reflexive and not semidualizing such that $A \not\simeq 0$, so the desired conclusion follows from Theorem 3.6. This uses the fact that if $A \simeq A_1 \times 0$, then A_1 is isomorphic in $\mathcal{D}(R)$ to a module and $A \cong A_1 \times 0$. □

REMARK 3.9. Other results for modules can be deduced from our results for complexes. We leave it as an exercise for the interested reader to formulate them.

We end with two consequences for integral domains that parallel our local results.

COROLLARY 3.10. *Assume that R is an integral domain, and let A be a homologically finite R-complex. Then the following conditions are equivalent:*

(i) *A is a semidualizing R-complex,*
(ii) *A is derived A-reflexive and $A \not\simeq 0$,*
(iii) *$\mathbf{R}\mathrm{Hom}_R(A, A)$ is derived A-reflexive and $A \not\simeq 0$ and*
(iv) *$0 \not\simeq A \in \mathcal{B}_A(R)$.*

Proof. (ii) $\Longrightarrow$ (i) Assume that A is derived A-reflexive and $A \not\simeq 0$. If A is not semidualizing, then Theorem 3.6 provides a non-trivial decomposition $R \cong R_1 \times R_2$, contradicting the assumption that R is a domain.

The remaining implications follow similarly, using Remark 3.2. □

The next result is proved like the previous one, using Theorem 3.7.

COROLLARY 3.11. *Assume that R is an integral domain, and let A be a homologically finite R-complex. Then $0 \not\simeq A \in \mathcal{A}_A(R)$ if and only if A is a tilting R-complex.*

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REFERENCES

1. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra* (Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969). MR 0242802 (39 #4129)

2. M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, vol. 1966/67, (Secrétariat mathématique, Paris, 1967). MR 37 #1435

3. M. Auslander and M. Bridger, *Stable module theory, Memoirs of the American Mathematical Society, vol. 94* (American Mathematical Society, Providence, R.I., 1969). MR 42 #4580

4. L. L. Avramov and H.-B. Foxby, Ring homomorphisms and finite Gorenstein dimension, *Proc. London Math. Soc.* **75**(3) (1997), 241–270. MR 98d:13014

5. L. L. Avramov, S. B. Iyengar and J. Lipman, Reflexivity and rigidity for complexes. I. Commutative rings, *Algebra Number Theory* **4**(1) (2010), 47–86. MR 2592013

6. L. W. Christensen, Semi-dualizing complexes and their Auslander categories, *Trans. Am. Math. Soc.* **353**(5) (2001), 1839–1883. MR 2002a:13017

7. E. E. Enochs, O. M. G. Jenda and J. Z. Xu, Foxby duality and Gorenstein injective and projective modules, *Trans. Amer. Math. Soc.* **348**(8) (1996), 3223–3234. MR 1355071 (96k:13010)

8. H.-B. Foxby, Gorenstein modules and related modules, *Math. Scand.* **31** (1972), 267–284. MR 48 #6094

9. H.-B. Foxby, Gorenstein dimensions over Cohen-Macaulay rings, in *Proceedings of the international conference on commutative algebra* (Osnabrück, Germany) (Bruns W., Editor) (Universität Osnabrück, 1994), 59–63.

10. A. Frankild and S. Sather-Wagstaff, Reflexivity and ring homomorphisms of finite flat dimension, *Commun. Algebra* **35**(2) (2007), 461–500. MR 2294611

11. A. Frankild and S. Sather-Wagstaff, The set of semidualizing complexes is a nontrivial metric space, *J. Algebra* **308**(1) (2007), 124–143. MR 2290914

12. A. J. Frankild, S. Sather-Wagstaff and A. Taylor, Relations between semidualizing complexes, *J. Commut. Algebra* **1**(3) (2009), 393–436. MR 2524860

13. E. S. Golod, *G*-dimension and generalized perfect ideals, Algebraic geometry and its applications, *Trudy Mat. Inst. Steklov.* **165** (1984), 62–66. MR 85m:13011

14. R. Hartshorne, *Residues and duality, Lecture Notes in Mathematics, vol.* 20 (Springer-Verlag, Berlin, 1966). MR 36 #5145

15. R. Hartshorne, *Local cohomology, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961* (Springer-Verlag, Berlin, 1967). MR 0224620 (37 #219)

16. S. Sather-Wagstaff, Complete intersection dimensions and Foxby classes, *J. Pure Appl. Algebra* **212**(12) (2008), 2594–2611. MR 2452313 (2009h:13015)

17. S. Sather-Wagstaff, *Bass numbers and semidualizing complexes, Commutative Algebra and its Applications* (Walter de Gruyter, Berlin, 2009), 349–381. MR 2640315

18. W. V. Vasconcelos, *Divisor theory in module categories, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]* (North-Holland Publishing Co., Amsterdam, 1974). MR 0498530 (58 #16637)

19. S. Yassemi, G-dimension, *Math. Scand.* **77**(2) (1995), 161–174. MR 97d:13017