MONOMIAL IDEALS: HOMEWORK 3

Exercise 1. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let I be an ideal of R. Prove that the following conditions are equivalent.

(i) I is a monomial ideal.
(ii) I is generated by monomials.
(iii) For each $f \in I$ each monomial occurring in f is in I.

Exercise 2. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let f, g and h be monomials in R.

(a) Show that, if $fh = gh$, then $f = g$.
(b) Show that, if $fX_i = gX_j$ for some $i \neq j$, then $f \in (X_j)R$ and $g \in (X_i)R$.

Exercise 3. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X, Y]$ in two variables. Set $J = (X^3, X^2Y, Y^3)R$ and $(\mathbf{X}) = (X, Y)R$.

(a) Verify that $J = (X^2, Y^3)R \cap (X^3, Y)R$.
(b) Verify that the monomials in $(J :_R (\mathbf{X})) \smallsetminus J$ are exactly XY^2 and X^2.

Exercise 4. (Fact II.3.12) Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let $J \subseteq R$ be a monomial ideal, and let $z_1, \ldots, z_n \in [J]$ be an irredundant generating sequence for J. Let $f, g \in R$ be monomials such that $f \neq 1_A$ and $z_1 = fg$. Prove that $g \notin J$.