MATH 724, FALL 2009, HOMEWORK 1
DUE FRIDAY 11 SEPTEMBER

Exercise 1. Let R be a commutative ring. Let M be an R-module and let $g\colon N \to N'$ be an R-module homomorphism.

(a) Prove that the map $\delta_N^M\colon N \to \mathrm{Hom}_R(\mathrm{Hom}_R(N, M), M)$ given by $\delta_N^M(n)(\psi) = \psi(n)$ is a well-defined R-module homomorphism.

(b) Prove that the following diagram commutes:

$$
\begin{array}{ccc}
N & \xrightarrow{\ \delta_N^M\ } & \mathrm{Hom}_R(\mathrm{Hom}_R(N, M), M) \\
{\scriptstyle g}\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Hom}_R(\mathrm{Hom}_R(g,M),M)} \\
N' & \xrightarrow{\ \delta_{N'}^M\ } & \mathrm{Hom}_R(\mathrm{Hom}_R(N', M), M).
\end{array}
$$

Exercise 2. Let R be a commutative ring, and let C be an R-module.

(a) Prove that the map $\chi_C^R\colon R \to \mathrm{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$ is a well-defined R-module homomorphism.

(b) Prove that $\mathrm{Ker}(\chi_C^R) = \mathrm{Ann}_R(C)$.

(c) Prove that if C is cyclic, then $C \cong R/\mathrm{Ann}_R(C)$.