MATH 724, FALL 2009, HOMEWORK 2
DUE FRIDAY 25 SEPTEMBER

Exercise 1. (This is Exercise IX.4.3 from the Homological Algebra Notes.) Let R be a commutative noetherian ring, and let M be a finitely generated R-module. Assume that $\dim(R)$ is finite. Prove that the following conditions are equivalent:

(i) $\mathrm{id}_R(M) < \infty$;
(ii) $\mathrm{id}_{U^{-1}R}(U^{-1}M) < \infty$ for each multiplicatively closed subset $U \subseteq R$;
(iii) $\mathrm{id}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for each prime ideal $\mathfrak{p} \subsetneq R$; and
(iv) $\mathrm{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subsetneq R$.

Exercise 2. Let R be a commutative ring, and let C be an R-module. Consider the following maps:

$f\colon \mathrm{Hom}_R(R, C) \to C$ given by $f(\psi) = \psi(1)$.
$\chi_C^R\colon R \to \mathrm{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$.
$\delta_R^C\colon R \to \mathrm{Hom}_R(\mathrm{Hom}_R(R, C), C)$ given by $\delta_R^C(r)(\psi) = \psi(r)$.
$\delta_C^C\colon C \to \mathrm{Hom}_R(\mathrm{Hom}_R(C, C), C)$ given by $\delta_C^C(c)(\phi) = \phi(c)$.

Prove that the following diagrams commute:

(1)
$$\begin{array}{ccc} R & \xrightarrow{\chi_C^R} & \mathrm{Hom}_R(C,C) \\ & \searrow^{\delta_R^C} & \cong \big\downarrow \mathrm{Hom}_R(f,C) \\ & & \mathrm{Hom}_R(\mathrm{Hom}_R(R,C),C) \end{array}$$

(2)
$$\begin{array}{ccc} C & \xrightarrow{\delta_C^C} & \mathrm{Hom}_R(\mathrm{Hom}_R(C,C),C) \\ \mathrm{id}_C \big\downarrow & & \big\downarrow \mathrm{Hom}_R(\chi_C^R,C) \\ C & \xleftarrow[\cong]{f} & \mathrm{Hom}_R(R,C). \end{array}$$