**MATH 724, FALL 2009, HOMEWORK 4**
**DUE FRIDAY 23 OCTOBER**

**Exercise 1.** (60 pts.) Let $R$ be a commutative ring, and let $M$ and $N$ be $R$-modules. The natural *evaluation map*

$$\xi_N^M \colon M \otimes_R \operatorname{Hom}_R(M,N) \to N$$

is the $R$-module homomorphism given by $\xi_N^M(m \otimes \psi) = \psi(m)$. The natural map

$$\gamma_N^M \colon N \to \operatorname{Hom}_R(M, M \otimes_R N)$$

is the $R$-module homomorphism given by $\gamma_N^M(n)(m) = m \otimes n$.

(a) (30 pts.) Prove that $\xi_N^M$ and $\gamma_N^M$ are well-defined $R$-module homomorphisms.

(b) (30 pts.) Let $f \colon N \to N'$ be an $R$-module homomorphism, and prove that the following diagram commutes:

$$\begin{array}{ccccc}
M \otimes_R \operatorname{Hom}_R(M,N) & \xrightarrow{\xi_N^M} & N & \xrightarrow{\gamma_N^M} & \operatorname{Hom}_R(M, M \otimes_R N) \\
\Big\downarrow{\scriptstyle M \otimes_R \operatorname{Hom}_R(M,f)} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \operatorname{Hom}_R(M, M \otimes_R f)} \\
M \otimes_R \operatorname{Hom}_R(M,N') & \xrightarrow{\xi_{N'}^M} & N' & \xrightarrow{\gamma_{N'}^M} & \operatorname{Hom}_R(M, M \otimes_R N')
\end{array}$$

**Exercise 2.** (40 pts.) Let $R$ be a commutative ring. Prove that the Auslander class $\mathcal{A}_R(R)$ and the Bass class $\mathcal{B}_R(R)$ both contain every $R$-module.

Extra credit: Let $C$ be a semidualizing $R$-module. If $\mathcal{A}_C(R)$ (or $\mathcal{B}_R(R)$) contains all $R$-modules, must $C$ be isomorphic to $R$? If so, why? If not, explain why not and provide conditions on $R$ that guarantee that the answer to the question is "yes".