MATH 720, Algebra I
Exercises 10
Due Fri 18 Nov

Exercise 1. Let R be a ring.

(a) Prove that given two exact sequences

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \qquad\qquad 0 \to C \xrightarrow{\gamma} D \xrightarrow{\delta} E$$

the following sequence is also exact:

$$A \xrightarrow{\alpha} B \xrightarrow{\gamma\circ\beta} D \xrightarrow{\delta} E.$$

We say that this sequence is obtained by "splicing" the given sequences at C.

(b) Assume that R has identity. Prove that given a unital R-module M, there is an exact sequence of R-module homomorphisms

$$\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

such that each F_i is a free R-module. Such a sequence is called a *free resolution* of M.

(c) (Bonus) Assume that R has identity. Prove that given a homomorphism of unital R-modules $f\colon M \to M'$, there is a homomorphism of exact sequences

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \xrightarrow{\partial_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{f}_1} & & \downarrow{\scriptstyle \widetilde{f}_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \xrightarrow{\partial'_2} & F'_1 & \xrightarrow{\partial'_1} & F'_0 & \xrightarrow{\partial'_0} & M' & \longrightarrow & 0
\end{array}$$

such that F_i and F'_i are free R-modules for all i.

Exercise 2. Let R be a ring, and consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc}
 & & M' & \xrightarrow{F'} & M & \xrightarrow{F} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & N' & \xrightarrow{G'} & N & \xrightarrow{G} & N''. & &
\end{array}$$

(a) Prove that there is an exact sequence

$$\operatorname{Ker}(f') \xrightarrow{\widetilde{F'}} \operatorname{Ker}(f) \xrightarrow{\widetilde{F}} \operatorname{Ker}(f'')$$

where $\widetilde{F'}$ and $\widetilde{F}$ are the maps from Exercises 6.2.

(b) Prove that there is an exact sequence

$$\operatorname{Coker}(f') \xrightarrow{\overline{G'}} \operatorname{Coker}(f) \xrightarrow{\overline{G}} \operatorname{Coker}(f'')$$

where $\overline{G'}$ and $\overline{G}$ are the maps from Exercises 6.2.

(c) (Snake Lemma, Bonus) Prove that there is an exact sequence

$$\operatorname{Ker}(f') \xrightarrow{\widetilde{F_1}} \operatorname{Ker}(f) \xrightarrow{\widetilde{G_1}} \operatorname{Ker}(f'') \xrightarrow{\eth} \operatorname{Coker}(f') \xrightarrow{\overline{F_0}} \operatorname{Coker}(f) \xrightarrow{\overline{G_0}} \operatorname{Coker}(f'').$$

(Hint: Define $\eth$ as follows. For each $m'' \in \mathrm{Ker}(f'')$, choose $m \in M$ such that $F(m) = m''$. Show that there is an element $n' \in N'$ such that $G'(n') = f(m)$. Define $\eth(m'') = \overline{n'} \in \mathrm{Coker}(f')$. Show that $\eth$ is well-defined by showing that it is independent of the choice of m and independent of the choice of n'. Then show that $\eth$ is an R-module homomorphism. Then show that the sequence is exact.)