MATH 720, Algebra I
Exercises 11–12
Due Fri 02 Dec

**Exercise 1.** Let $R$ be a ring and consider an exact sequence of $R$-modules:

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

Prove that $M$ is noetherian if and only if $M'$ and $M''$ are noetherian.

**Exercise 2.** Let $R$ be a ring with identity. Prove that $R$ is a noetherian ring if and only if every finitely generated unital $R$-module is noetherian.

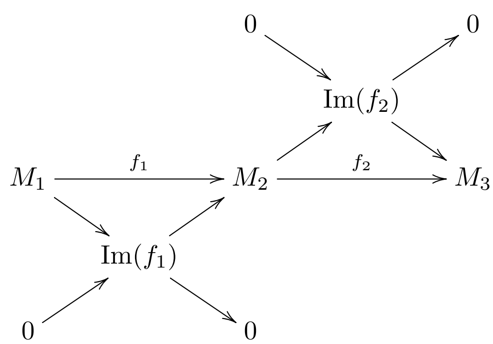
**Exercise 3.** Let $R$ be a principal ideal domain, and let $M$ be a finitely generated unital $R$-module.

(a) Prove that there are integers $n \geqslant k \geqslant 1$ and elements $d_1, \ldots, d_k \in R$ such that $M \cong R/d_1R \oplus \cdots \oplus R/d_kR \oplus R^{n-k}$ and $d_i \big| d_{i+1}$ for $i = 1, \ldots, k-1$.

(b) (Bonus) State and prove a uniqueness result for the integers $n$, $k$ and the elements $d_1, \ldots, d_k \in R$.

**Exercise 4.** $R$ is a ring with identity and $P$ is a unital $R$-module. Prove that $P$ is projective if and only if for each epimiorphism $f \colon M \to N$ of unitary $R$-modules, the induced map $f_* \colon \operatorname{Hom}_R(P, M) \to \operatorname{Hom}_R(P, N)$ is surjective.

[Hint for the implication $\Longleftarrow$ : Let $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ be an exact sequence. Show that there is a commutative diagram of unital $R$-module homomorphisms




such that each diagonal sequence is exact. Apply $\operatorname{Hom}_R(P, -)$ to this diagram. Conclude that each diagonal sequence in the resulting diagram is exact. Deduce that the horizontal sequence in the resulting diagram is exact.]

**Exercise 5.** $R$ is a ring with identity and $P$ is a unital $R$-module. If $P$ is finitely generated and projective, then there is a finitely generated projective unitary $R$-module $Q$ such that $P \oplus Q$ is free of finite rank.