MATH 720, Algebra I
Exercises 13
Due Fri 09 Dec

Let n be a positive integer.

Exercise 1. Let $A \in \mathcal{M}_{n\times n}(K)$, and consider $V = K^n$ as a $K[X]$-module where $Xv = Av$ for all $v \in V$. Let $f_1, \ldots, f_t \in K[X]$ be irreducible and monic, and let $e_1, \ldots, e_t \geqslant 1$ such that $V \cong \oplus_{i=1}^t K[X]/(f_i^{e_i})$. Prove that the following conditions are equivalent.

(i) A is diagonalizable, i.e., A is similar to a diagonal matrix D.
(ii) V has a matrix consisting of eigenvectors for A.
(iii) for $i = 1, \ldots, t$ we have $e_i = 1 = \deg(f_i)$.

Exercise 2.

- Let R be a commutative ring with identity, and let $i, j \in \mathbb{N}$ such that $1 \leqslant i, j \leqslant n$.
- Let $e_1, \ldots, e_{n-1} \in R^{n-1}$ and $\epsilon_1, \ldots, \epsilon_n \in R^n$ be the standard bases.
- Let $f\colon (R^{n-1})^{n-1} \to R$ be the unique multilinear anti-symmetric function such that $f(e_1, \ldots, e_{n-1}) = 1$.
- Let $g\colon (R^n)^n \to R$ be defined by "expanding along the ith row" using f to take the appropriate $(n-1) \times (n-1)$ sub-determinants.
- Let $h\colon (R^n)^n \to R$ be defined by "expanding along the jth column" using f to take the appropriate $(n-1) \times (n-1)$ sub-determinants.

Prove that g and h are multilinear anti-symmetric functions such that $g(\epsilon_1, \ldots, \epsilon_n) = 1 = h(\epsilon_1, \ldots, \epsilon_n)$.