MATH 720, Algebra I
Exercises 2
Due Fri 09 Sep

Exercise 1. Let G be a group with operation written multiplicatively, and let R be a ring with identity. Let $R[G]$ be the group $R^{(G)}$, written additively. The elements of $R[G]$ are finite formal sums $\sum_{g\in G} r_g g$ with the $r_g \in R$, and addition is given by

$$(\textstyle\sum_{g\in G} r_g g) + (\sum_{g\in G} s_g g) = \sum_{g\in G}(r_g + s_g)g.$$

Define multiplication on $R[G]$ by the formula

$$(\textstyle\sum_{g\in G} r_g g)(\sum_{h\in G} s_h h) = \sum_{g,h\in G}(r_g s_h)(gh).$$

(a) Prove that $R[G]$ is a ring with identity $1_{R[G]} = 1_R e_G$.
(b) Prove that $R[G]$ is commutative if and only if R is commutative and G is abelian.

Exercise 2 (Binomial Theorem)**.** Let R be a ring with identity and let $n \in \mathbb{N}$.

(a) Let $a, b \in R$ such that $ab = ba$. Prove that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.
(b) Let $a_1, \ldots, a_m \in R$ such that $a_i a + j = a_j a_i$ for all i, j. Prove that

$$(a_1 + \cdots + a_m)^n = \sum_{k_1+\cdots k_m=n} \frac{n!}{k_1! \cdots + k_m!} a_1^{k_1} \cdots a_m^{k_m}.$$