MATH 720, Algebra I
Exercises 3
Due Fri 16 Sep

Exercise 1. Let R be a ring, and fix an integer $n \geqslant 2$. Let $S_1, \ldots, S_n \subseteq R$ be subsets, and set $I_j = (S_j)R$ for $j = 1, \ldots, n$.

(a) Prove that $\sum_j I_j = (\cup_j S_j)R$.

(b) Prove that if R is commutative, then $\prod_j I_j = (\{a_1 \cdots a_n \mid a_j \in S_j, j = 1, \ldots, n\})R$.

(c) Prove that if R is commutative, then any finite product of principal ideals is principal.

Exercise 2. Let $\{R_\alpha\}_{\alpha \in A}$ be a set of rings, and for each $\lambda \in \Lambda$ fix an ideal $I_\lambda \leqslant R_\lambda$.

(a) Prove that $\bigoplus_\lambda I_\lambda$ is an ideal of $\bigoplus_\lambda R_\lambda$ and that there is a ring isomorphism

$$(\textstyle\bigoplus_\lambda R_\lambda)/(\bigoplus_\lambda I_\lambda) \cong \bigoplus_\lambda (R_\lambda/I_\lambda).$$

(b) Prove that $\prod_\lambda I_\lambda$ is an ideal of $\prod_\lambda R_\lambda$. and that there is a ring isomorphism

$$(\textstyle\prod_\lambda R_\lambda)/(\prod_\lambda I_\lambda) \cong \prod_\lambda (R_\lambda/I_\lambda).$$

(c) Prove that $\bigoplus_\lambda I_\lambda$ is an ideal of $\prod_\lambda R_\lambda$. Deduce that $\bigoplus_\lambda R_\lambda$ is an ideal of $\prod_\lambda R_\lambda$.

(d) Given elements $(r_\lambda), (s_\lambda) \in \prod_\lambda R_\lambda$, prove that the cosets $\overline{(r_\lambda)}$ and $\overline{(s_\lambda)}$ in $(\prod_\lambda R_\lambda)/(\bigoplus_\lambda R_\lambda)$ are equal if and only if $r_\lambda = s_\lambda$ for all but finitely many $\lambda \in \Lambda$.

Exercise 3. Prove or give a counterexample: given rings R_1 and R_2 with identity, every ideal of $R_1 \times R_2$ is of the form $I_1 \oplus I_2$ for some ideals $I_1 \leqslant R_1$ and $I_2 \leqslant R_2$.