MATH 720, Algebra I
Exercises 4
Due Fri 23 Sep

Exercise 1. Let G be a group with operation written multiplicatively, and let R be a commutative ring with identity. Let $R[G]$ be the group ring (from Exercises 3). Let $f\colon G \to H$ be a multiplicative group homomorphism.

(a) Prove that the function $F\colon R[G] \to R[H]$ given by $F(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g f(g)$ is a well-defined homomorphism of rings with identity.

(b) Prove that $\mathrm{Ker}(F)$ is the set of all elements $\sum_{g\in G} r_g g \in R[G]$ such that for each $g \in G$ one has $\sum_{h\in\mathrm{Ker}(f)} r_{gh} = 0_R$.

(c) Let N be a normal subgroup of G, and let K be the set of all elements $\sum_{g\in G} r_g g \in R[G]$ such that for each $g \in G$ one has $\sum_{h\in N} r_{gh} = 0_R$. Prove that K is an ideal of $R[G]$.

(d) Continue with the notation of part (c). Prove that the ideal K is maximal if and only if $N = G$ and R is a field.

Exercise 2. Let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a set of non-zero commutative rings with identity, and for each $\lambda \in \Lambda$ fix an ideal $I_\lambda \leqslant R_\lambda$.

(a) Prove that $\prod_\lambda R_\lambda$ is an integral domain if and only if Λ contains a single element λ, and R_λ is an integral domain.

(b) Prove that $\prod_\lambda I_\lambda$ is a prime ideal of $\prod_\lambda R_\lambda$ if and only if there is an element $\mu \in \Lambda$ such that $I_\lambda = R_\lambda$ for all $\lambda \neq \mu$ and I_μ is a prime ideal of R_μ.

Exercise 3. Let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a set of non-zero commutative rings with identity, and assume that Λ is an infinite set. Prove that $\prod_\lambda R_\lambda$ has a maximal ideal containing $\bigoplus_\lambda R_\lambda$. Use this to prove that $\prod_\lambda R_\lambda$ has a maximal ideal that is not of the form $\prod_\lambda I_\lambda$ or $\bigoplus_\lambda I_\lambda$. (Note that Exercise set 3 shows that this is very different from the case where Λ is finite.)