MATH 720, Algebra I
Exercises 5
Due Fri 07 Oct

Let R be a commutative ring with identity.

Definition 1. An element $r \in R$ is *nilpotent* if there is an integer $n \geqslant 1$ such that $r^n = 0$.

Exercise 1. Prove that a polynomial in $R[X_1, \ldots, X_n]$ is nilpotent if and only if all its coefficients are nilpotent. (Hint: Induction on n.)

Exercise 2. Prove that a polynomial $\sum_{\mathbf{a} \in \mathbb{N}^n} r_{\mathbf{a}} X_1^{a_1} \cdots X_n^{a_n} \in R[X_1, \ldots, X_n]$ is a unit if and only if $r_{\mathbf{0}}$ is a unit in R and $r_{\mathbf{a}}$ is nilpotent for all $\mathbf{a} \neq \mathbf{0}$. (Hint: Induction on n.)

Exercise 3. Let $I \subseteq R$ be an ideal. Prove that $I[X_1, \ldots, X_n]$ is an ideal of $R[X_1, \ldots, X_n]$ such that $R[X_1, \ldots, X_n]/I[X_1, \ldots, X_n] \cong (R/I)[X_1, \ldots, X_n]$.