MATH 720, Algebra I
Exercises 6
Due Fri 21 Oct

Exercise 1. Prove the Rational Root Theorem: Let R be a UFD with $K = \mathrm{Q}(R)$. Let $f = a_0 + a_1x + \cdots + a_dx^d \in R[x]$ and assume that $r = a/b \in K$ is a root of f with $\gcd(a, b) = [1]$. Then $a\big|a_0$ and $b\big|a_d$. In particular, if f is monic, then $r \in R$.

Exercise 2. Let R be a ring, and consider the following commutative diagram of R-module homomorphisms:

$$\begin{CD} M @>f>> N \\ @V{\phi}VV @VV{\psi}V \\ M' @>{f'}>> N'. \end{CD}$$

(a) Prove that there are unique R-module homomorphisms $\widetilde{\phi}\colon \mathrm{Ker}(f) \to \mathrm{Ker}(f')$ and $\overline{\psi}\colon N/\mathrm{Im}(f) \to N'/\mathrm{Im}(f')$ making the following diagram commute:

$$\begin{CD} \mathrm{Ker}(f) @>i>> M @>f>> N @>\pi>> N/\mathrm{Im}(f) \\ @V{\widetilde{\phi}}VV @V{\phi}VV @VV{\psi}V @VV{\overline{\psi}}V \\ \mathrm{Ker}(f') @>{i'}>> M' @>{f'}>> N' @>{\pi'}>> N'/\mathrm{Im}(f'). \end{CD}$$

[Note: $N/\mathrm{Im}(f)$ is called the "cokernel" of f and is denoted $\mathrm{Coker}(f)$. As the kernel characterizes monomorphisms (f is 1-1 if and only if $\mathrm{Ker}(f) = 0$), the cokernel characterizes epimorphisms: f is onto if and only if $\mathrm{Coker}(f) = 0$.]

(b) Prove that if ϕ is 1-1, then so is $\widetilde{\phi}$.

(c) Prove that if ψ is onto, then so is $\overline{\psi}$.

(d) Prove that if ϕ is onto and ψ is 1-1, then $\widetilde{\phi}$ is onto and $\overline{\psi}$ is 1-1.