MATH 720, Algebra I
Exercises 7
Due Fri 21 Oct

Exercise 1. Let R be a commutative ring, let $U \subseteq R$ be a subset, and let I be an ideal of R. Let M be an R-module, and let $N \subseteq M$ be an R-submodule.

(a) Prove that the set

$$(N :_M U) = \{m \in M \mid um \in N \text{ for all } u \in U\}$$

is an R-submodule of M.

(b) Prove that if R has identity, then there is an isomorphism of R-modules

$$\operatorname{Hom}_R(R/I, M) \cong (0 :_M I).$$

Exercise 2. Let R be a ring, let I be an ideal of R, and let M be an R-module. Assume that $am = 0$ for all $a \in I$ and for all $m \in M$. Prove that M has a well-defined R/I-module structure defined by $(r + I)m := rm$.

Exercise 3. Let R be a commutative ring, and let M be an R-module. Let n be a positive integer, and let $x_1, \ldots, x_n \in R$. Prove that M is an $R[X_1, \ldots, X_n]$-module using the scalar multiplication

$$(\textstyle\sum_{\underline{a} \in \mathbb{N}} r_{\underline{a}} \underline{X}^{\underline{a}})m := \textstyle\sum_{\underline{a} \in \mathbb{N}} r_{\underline{a}} \underline{x}^{\underline{a}} m.$$

In other words, we define $X_i m := x_i m$ and extend this to an $R[X_1, \ldots, X_n]$-module structure in the natural way.