MATH 720, Algebra I
Exercises 8
Due Fri 28 Oct

Exercise 1. Let R be a ring with identity, and let M be a unital R-module. For each $m \in M$, set

$$\operatorname{Ann}_R(m) = (0 :_R m) = \{r \in R \mid rm = 0\}.$$

Also, set

$$\operatorname{Ann}_R(M) = (0 :_R M) = \{r \in R \mid rm = 0 \text{ for each } m \in M\}.$$

(a) Let $X \subseteq M$ be a generating set for M, and prove that

$$\operatorname{Ann}_R(M) = \bigcap_{x \in X} \operatorname{Ann}_R(x).$$

(b) Prove that if M is cyclic, then $M \cong R/\operatorname{Ann}_R(M)$.
(c) Assume that R is commutative, and let $\mathfrak{m} \subset R$ be a maximal ideal. Assume that M is non-zero and cyclic such that $rx = 0$ for all $r \in \mathfrak{m}$ and all $x \in M$. Prove that $M \cong R/\mathfrak{m}$.

Exercise 2. Let R be an integral domain that is not a field, and let $Q(R)$ denote the quotient field of R. Prove that $Q(R)$ is not finitely generated as an R-module, using the following steps.

(a) Suppose by way of contradiction that $Q(R)$ is finitely generated.
(b) Prove that $Q(R)$ is cyclic.
(c) Prove that the natural map $R \to Q(R)$ is an isomorphism.
(d) Derive a contradiction.