MATH 720, Algebra I
Exercises 9
Due Fri 18 Nov

Exercise 1. Let k be a field, and let V be a finite dimensional vector space over k. Prove that, given two subspaces $A, B \subseteq V$, we have

$$\dim_k(A + B) = \dim(A) + \dim(B) - \dim(A \cap B).$$

Exercise 2. Let R be a commutative ring with identity. Let F be a free R-module of rank m with basis $f_1, \ldots, f_m \in F$. Let G be a free R-module of rank n with basis $g_1, \ldots, g_n \in G$. Let $e_1, \ldots, e_m \in R^m$ be the standard basis, and let $\epsilon_1, \ldots, \epsilon_n \in R^n$ be the standard basis.

The universal mapping property for free modules implies that for $i = 1, \ldots, n$ there is a unique R-module homomorphism $f_i^* \colon F \to R$ such that

$$f_i^*(f_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

(a) Prove that $\mathrm{Hom}_R(F, R)$ is a free R-module of rank n with basis $f_1^*, \ldots, f_n^*$. This is called the *dual basis* for F^*.[1] Conclude that there is a unique isomorphism $\psi_F \colon F^* \to R^m$ such that $\psi(f_i^*) = e_i$ for each i.

(b) Prove that if $\phi \colon F \to G$ is the R-module homomorphism represented by the matrix A with respect to the f_i's and g_j's, then there is a commutative diagram

$$\begin{array}{ccc} G^* & \xrightarrow{\phi^*} & F^* \\ {\scriptstyle \psi_G}\downarrow{\scriptstyle \cong} & & {\scriptstyle \psi_F}\downarrow{\scriptstyle \cong} \\ R^n & \xrightarrow{A^T} & R^m. \end{array}$$

In other words, $\phi^* \colon G^* \to F^*$ is represented by the matrix transpose A^T with respect to the g_j^*'s and f_i^*'s.

(c) (Bonus) Prove that the map $\delta_F \colon F \to F^{**}$ given by $\delta_F(f)(\alpha) = \alpha(f)$ is a well-defined R-module isomorphism. Here $F^{**} := (F^*)^*$.

[1] We frequently write $F^* = \mathrm{Hom}_R(F, R)$. This fits with the notation ϕ^* from class: given a homomorphism of free R-modules $\phi \colon F \to G$, the induced map ϕ^* maps $G^* \to F^*$.