MATH 720, Algebra I
Exam 1
Due Fri 30 Sep

Let R be a commutative ring with identity.

Exercise 1. Let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a non-empty set of commutative rings with identity. For each $\mu \in \Lambda$ let $p_\mu \colon \prod_{\lambda\in\Lambda} R_\lambda \to R_\mu$ be defined as $p_\mu((r_\lambda)) := r_\mu$.

(a) Prove that for each $\mu \in \Lambda$ the function $p_\mu \colon \prod_{\lambda\in\Lambda} R_\lambda \to R_\mu$ is an epimorphism of commutative rings with identity.
(b) Let $\{f_\lambda \colon R \to R_\lambda\}_{\lambda\in\Lambda}$ be a set of homomorphisms of commutative rings with identity. Prove that there is a unique homomorphism $F \colon R \to \prod_{\lambda\in\Lambda} R_\lambda$ such that for each $\mu \in \Lambda$ the composition $p_\mu \circ F$ is f_μ.

Exercise 2. Let I be an ideal of R. Let $R \ltimes I$ be the additive abelian group $R \oplus I$ with the following multiplication: for all $(r, i), (r', i') \in R \ltimes I$ we set $(r, i)(r', i') := (rr', ri' + r'i)$.

(a) Prove that $R \ltimes I$ is a commutative ring with identity under these operations.
(b) Prove that the map $g \colon R \to R \ltimes I$ given by $g(r) = (r, 0)$ is a monomorphism of rings with identity.
(c) Prove that the subset $0 \oplus I \subseteq R \ltimes I$ is an ideal of R such that $(R \ltimes I)/(0 \oplus I) \cong R$.
(d) Prove that $(0 \oplus I)^2 = 0$.
(e) Prove that if I is generated (as an ideal of R) by a set $S \subseteq R$, then $0 \oplus I$ is generated (as an ideal of $R \ltimes I$) by the set $\{(0, s) \in R \ltimes I \mid s \in S\}$.

Definition 1. Let I be an ideal of R. The *radical* of I is the set

$$\operatorname{rad}(I) = \{x \in R \mid \text{there is an integer } n \geqslant 1 \text{ such that } x^n \in I\}.$$

Exercise 3. Let I and J be ideals of R.

(a) Prove that $\operatorname{rad}(I)$ is an ideal of R such that $I \subseteq \operatorname{rad}(I) = \operatorname{rad}(\operatorname{rad}(I))$.
(b) Prove that if $I \subseteq J$, then $\operatorname{rad}(I) \subseteq \operatorname{rad}(J)$.
(c) Prove that $\operatorname{rad}(I) = R$ if and only if $I = R$.
(d) Prove that if I is finitely generated and $I \subseteq \operatorname{rad}(J)$, then there is an integer $q \geqslant 1$ such that $I^q \subseteq J$.
(e) Assume that R is a unique factorization domain, and let u be a unit of R. Let $p_1, \ldots, p_n$ be primes of R such that for all i, j such that $1 \leqslant i < j \leqslant n$ the elements p_i and p_j are not associates. Given integers $e_1, \ldots, e_n \geqslant 1$ prove that $\operatorname{rad}((up_1^{e_1} \cdots p_n^{e_n})R) = (p_1 \cdots p_n)R$.
(f) Find an example of a commutative ring R with identity and two ideals I and J such that $\operatorname{rad}(I) = \operatorname{rad}(J)$ but $I \not\subseteq J$ and $J \not\subseteq I$.