MATH 720, Algebra I
Exam 2
Due Fri 04 Nov

Let R be a commutative ring with identity, and let M be a unital R-module.

Exercise 1. (a) Prove that $\mathrm{Hom}_R(M, M)$ is a ring with identity, where addition is performed point-wise, and multiplication is composition of functions.

(b) Prove that if M is cyclic, then $\mathrm{Hom}_R(M, M)$ is a commutative ring with identity.

(c) Prove that the conclusion of part (b) can fail if M is not cyclic.

(d) Prove that M is a $\mathrm{Hom}_R(M, M)$-module with scalar multiplication given by $f \cdot m = f(m)$.

(e) For each $r \in R$, the function $\mu_r \colon M \to M$ given by $\mu_r(m) = rm$ is an R-module homomorphism. Prove that the function $\chi \colon R \to \mathrm{Hom}_R(M, M)$ given by $\chi(r) := \mu_r$ is a homomorphism of rings with identity.

(f) Prove that the R-module structure on M given by restriction of scalars along χ is the same as the original R-module structure on M.

Exercise 2. Let $I \subseteq R$ be an ideal, and set

$$\begin{aligned}\Gamma_I(M) &= \cup_{n=1}^{\infty}(0 :_M I^n) \\ &= \{m \in M \mid \text{there is an integer } n \geqslant 1 \text{ such that } am = 0 \text{ for all } a \in I^n\}.\end{aligned}$$

(a) Prove that $\Gamma_I(M)$ is a submodule of M.

(b) Prove that, for each homomorphism $f \colon M \to N$ of unital R-modules, the function $\Gamma_I(f) \colon \Gamma_I(M) \to \Gamma_I(N)$ given by $\Gamma_I(f)(m) = f(m)$ is a well-defined R-module homomorphism.

(c) Prove that if $f \colon M \to N$ is a monomorphism of unital R-modules, then $\Gamma_I(f)$ is a monomorphism.

(d) Prove or give a counterexample: If $f \colon M \to N$ is a homomorphism of unital R-modules, such that $\Gamma_I(f)$ is a monomorphism, then f is a monomorphism.