MATH 720, Algebra I
Exam 3
Due Fri 16 Dec

Let R be a commutative ring with identity.

Exercise 1 (Cramer's Rule)**.** Consider the following system of n linear equations in n unknowns

$$\sum_{j=1}^{n} a_{1,j} x_j = b_1$$

$$\vdots$$

$$\sum_{j=1}^{n} a_{n,j} x_j = b_n$$

where the elements $a_{i,j}$ and b_i are in R. Let $A = (a_{i,j})$, and assume that $\det(A)$ is a unit in R. Prove that the system above has a unique solution and that the solution is given by the formula $x_i = \det(A)^{-1} \det(A_i)$ where A_i is the matrix formed by replacing the ith column of A by the column vector $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

Exercise 2. Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. We say that A is *nilpotent* if there is a natural number $q \geqslant 1$ such that $A^q = 0$. Prove that the following conditions are equivalent:

(i) A is nilpotent.
(ii) every eigenvalue of A is 0.
(iii) Prove that if A is nilpotent, then $A^n = 0$.

Exercise 3. Let M and N be unital R-modules. Assume that there are integers $m, n \geqslant 0$ and an exact sequence

$$R^m \to R^n \to M \to 0$$

of unital R-modules. Prove that, if N is noetherian, then $\operatorname{Hom}_R(M, N)$ is a noetherian R-module.

Exercise 4. Assume that R is noetherian, and let M and N be finitely generated unital R-modules. Prove that $\operatorname{Hom}_R(M, N)$ is a finitely generated R-module.

Exercise 5. Consider the following conditions:

(i) R is a field.
(ii) Every unital R-module is free.
(iii) Every unital R-module is projective.
(iv) Every short exact sequence $0 \to A \to B \to C \to 0$ of unital R-modules splits.

Complete the following:

(a) Prove the implications (i) $\iff$ (ii) $\implies$ (iii) $\iff$ (iv).
(b) Provide a counterexample to the implication (iii) $\implies$ (ii). (Be sure to prove that your counterexample has the desired properties.)