MATH 790, FALL 2011, HOMEWORK 1
DUE FRIDAY 02 SEPTEMBER

Exercise 1. Let R be a commutative ring. Let $g\colon N \to N'$ be an R-module homomorphism and let M be an R-module.

(a) Prove that the map $\delta_N^M\colon N \to \operatorname{Hom}_R(\operatorname{Hom}_R(N,M),M)$ given by $\delta_N^M(n)(\psi) = \psi(n)$ is a well-defined R-module homomorphism.
To be clear:
For each $n \in N$ define $\delta_N^M(n)\colon \operatorname{Hom}_R(N,M) \to M$ by the formula $\psi \mapsto \psi(n)$.
Prove that $\delta_N^M(n)$ is an R-module homomorphism.
Next, prove that the map $N \to \operatorname{Hom}_R(\operatorname{Hom}_R(N,M),M)$ given by $n \mapsto \delta_N^M(n)$ is a well-defined R-module homomorphism.

(b) Prove that the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{\delta_N^M} & \operatorname{Hom}_R(\operatorname{Hom}_R(N,M),M) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_R(\operatorname{Hom}_R(g,M),M)} \\ N' & \xrightarrow{\delta_{N'}^M} & \operatorname{Hom}_R(\operatorname{Hom}_R(N',M),M). \end{array}$$

Exercise 2. Let R be a commutative ring, and let C be an R-module.

(a) Prove that the map $\chi_C^R\colon R \to \operatorname{Hom}_R(C,C)$ given by $\chi_C^R(r)(c) = rc$ is a well-defined R-module homomorphism.

(b) Prove that $\operatorname{Ker}(\chi_C^R) = \operatorname{Ann}_R(C)$.

(c) Prove that if C is cyclic, then $C \cong R/\operatorname{Ann}_R(C)$.