MATH 790, FALL 2011, HOMEWORK 11–12
DUE FRIDAY 02 DECEMBER

Exercise 1. Let R be a noetherian ring, and let C be an R-module. Consider an exact sequence of R-module homomorphisms

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0.$$

(a) Prove that if $M_1, M_3 \in \mathcal{B}_C(R)$, then $M_2 \in \mathcal{B}_C(R)$.
(b) Prove that if C is semidualizing and $M_1, M_2 \in \mathcal{B}_C(R)$, then $M_3 \in \mathcal{B}_C(R)$.

Exercise 2. (Bonus) Let R be a noetherian ring.

(a) Prove that $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module.
(b) Prove that $\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is a faithfully injective R-module.

Exercise 3. Let R be a noetherian ring, and let C be a finitely generated R-module. Prove that the following conditions are equivalent:

(i) The R-module C is semidualizing;
(ii) The class $\mathcal{B}_C(R)$ contains every R-module of finite injective dimension;
(iii) The class $\mathcal{B}_C(R)$ contains every injective R-module; and
(iv) The class $\mathcal{B}_C(R)$ contains a faithfully injective R-module.

(You may assume that R has a faithfully injective module, by Exercise 2.)

Exercise 4. Let R be a noetherian ring, and let C be a semidualizing R-module. Prove that an R-module M is in $\mathcal{B}_C(R)$ if and only if $\operatorname{Hom}_R(C, M)$ is in $\mathcal{A}_C(R)$.