MATH 790, FALL 2011, HOMEWORK 13 (OPTIONAL)
DUE FRIDAY 09 DECEMBER

Definition 1. Let R be a commutative ring. An element $e \in R$ is *idempotent* if $e^2 = e$. (For instance, 0 and 1 are both idempotent.)

Exercise 2. Let R be a commutative local ring. Prove that the only idempotent elements of R are 0 and 1.

Exercise 3. Let $R_1, \dots, R_n$ be commutative rings. Prove that $\operatorname{Spec}(R_1 \times \cdots \times R_n)$ is the disjoint union of open (and closed) subspaces X_i, where X_i is homeomorphic to $\operatorname{Spec}(R_i)$.

Exercise 4. Let R be a commutative ring. Prove that the following conditions are equivalent.

(i) $\operatorname{Spec}(R)$ is disconnected.
(ii) There are nonzero commutative rings R_1 and R_2 such that $R \cong R_1 \times R_2$.
(iii) R contains an idempotent element $e \neq 0, 1$.

Conclude that if R is local or an integral domain, then $\operatorname{Spec}(R)$ is connected.