MATH 790, FALL 2011, HOMEWORK 3
DUE FRIDAY 23 SEPTEMBER

Let R be a commutative noetherian ring. This homework set uses the very famous and useful

Lemma 1 (Nakayama's Lemma)**.** *Let M be a finitely generated R-module. If $M/\mathfrak{m}M = 0$ for each maximal ideal $\mathfrak{m}$, then $M = 0$.*

Exercise 2. Let M and C be finitely generated R-modules such that

$$M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C).$$

Use the following steps to prove that the natural biduality map

$$\delta_M^C \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$$

is an isomorphism.

item0 (a) Let $f \colon N \to N'$ be an R-module homomorphism and set $\operatorname{Coker}(f) = N' / \operatorname{Im}(f)$. Prove that f is an epimorphism if and only if $\operatorname{Coker}(f) = 0$.

item1 (b) Let X and Y be finitely generated R-modules. Prove that if $X \oplus Y \cong Y$, then $X = 0$. [Hint: The modules $X/\mathfrak{m}X$ and $Y/\mathfrak{m}Y$ are finite dimensional vector spaces such that $(X/\mathfrak{m}X) \oplus (Y/\mathfrak{m}Y) \cong (Y/\mathfrak{m}Y)$, so $X/\mathfrak{m}X = 0$. Now apply Nakayama's Lemma. Note that this fails if Y is not finitely generated. Of course, if Y is finitely generated and $X \oplus Y \cong Y$, then X is finitely generated.]

item3 (c) Let N be an R-module. Prove that the composition

$$\operatorname{Hom}_R(\delta_N^C, C) \circ \delta_{\operatorname{Hom}_R(N,C)}^C$$

is the identity $\operatorname{id}_{\operatorname{Hom}_R(N,C)} \colon \operatorname{Hom}_R(N, C) \to \operatorname{Hom}_R(N, C)$.

item4 (d) Let N be an R-module. Conclude from part (c) that the homomorphism

$$\delta_{\operatorname{Hom}_R(N,C)}^C \colon \operatorname{Hom}_R(N, C) \to \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(N, C), C), C)$$

is a split monomorphism, so there is a finitely generated R-module $X = \operatorname{Coker}(\delta_{\operatorname{Hom}_R(N,C)}^C)$ such that

$$\operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(N, C), C), C) \cong X \oplus \operatorname{Hom}_R(N, C).$$

item5 (e) Set $N = \operatorname{Hom}_R(M, C)$ and use Step (d) to explain the final isomorphism in the next sequence where $X = \operatorname{Coker}(\delta_{\operatorname{Hom}_R(N,C)}^C)$:

$$\begin{aligned} M &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C) \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C), C), C) \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(N, C), C), C) \\ &\cong X \oplus \operatorname{Hom}_R(N, C) \\ &= X \oplus M. \end{aligned}$$

Use Step (b) to conclude that $X = 0$. Use Step (a) to conclude that $\delta_{\operatorname{Hom}_R(N,C)}^C$ is an epimorphism. Use Step (d) to conclude that $\delta_{\operatorname{Hom}_R(N,C)}^C = \delta_M^C$ is an isomorphism.